ENCYCLOPAEDIA OF VEDIC MATHEMATICS

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SHYAM PRIYA

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Vedic mathematics is based on sixteen Sutras and some sub-Sutras which provide links through mathematics: the word 'Sutra' means 'thread'. Vedic scheme is not simply a collection of rapid methods; it is a system, a unified approach, which can be swiftly learnt. Vedic Mathematics extensively exploits the properties of numbers in every practical applications, particularly in the field of computation. It makes available a whole range of methods ideally suited to these properties. After a detailed introduction to Vedic Mathematics, this book shows applications of Vertically and Crosswise from basic calculations to the evaluation of logarithms, exponentials, trigonometrical functions and the solution of simultaneous, transcendental, polynomial and differential equations. The methods given in this book are not intended to be a complete or thorough treatment of the topics they deal with. All of the ideas can probably be developed further or applied in other areas, and all can doubtless be improved upon.

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Preface

Vedic Mathematics contains many examples of striking methods of calculation and there is a remarkable coherence to the system which makes it very attractive. Also the Vedic system itself suggests a kind of approach that involves going directly to the answer. Vedic mathematics is based on sixteen Sutras and some sub-Sutras which provide links through mathematics: the word 'Sutra' means 'thread'.

Vedic scheme is not simply a collection of rapid methods; it is a system, a unified approach, which can be swiftly learnt. Vedic Mathematics extensively exploits the properties of numbers in every practical applications, particularly in the field of computation. It makes available a whole range of methods ideally suited to these properties. After a detailed introduction to Vedic Mathematics, this book shows applications of Vertically and Crosswise from basic calculations to the evaluation of logarithms, exponentials, trigonometrical functions and the solution of simultaneous, transcendental, polynomial and differential equations. The methods shown are not only exceedingly efficient and new but have a unity and coherence which renders them easy to master and enjoyable to execute.

The methods given in this book are not intended to be a complete or thorough treatment of the topics they deal with. All of the ideas can probably be developed further or applied in other areas, and all can doubtless be improved upon. The mathematician will also observe a certain lack of rigor as an attempt has been made to make the material intelligible to as wide a readership as possible.

—***Shyam Priya***

1

The Magic of Vedic Maths

What is Vedic Math?

What does mathematics have to do with Hinduism? Well, just as the basic principles of Hinduism lie in the Vedas, so do the roots of mathematics. The Vedas, written around 1500-900 BCE, are ancient Indian texts containing a record of human experience and knowledge. Thousands of years ago, Vedic mathematicians authored various theses and dissertations on mathematics. It is now commonly believed and widely accepted that these treatises laid down the foundations of algebra, algorithm, square roots, cube roots, various methods of calculation, and the concept of zero.

Vedic Mathematics

"Vedic Mathematics" is the name given to the ancient system of mathematics, or, to be precise, a unique technique of calculations based on simple rules and principles, with which any mathematical problem - be it arithmetic, algebra, geometry or trigonometry - can be solved, hold your breath, orally!

Sutras: Natural Formulae

The system is based on 16 Vedic sutras or aphorisms, which are actually word-formulae describing natural ways of solving a whole range of mathematical problems. Some examples of sutras are "By one more than the one before", "All from 9 & the last from 10", and "Vertically & Crosswise". These 16 one-line formulae originally written in Sanskrit, which can be easily memorized, enables one to solve long mathematical problems quickly.

Why Sutras?

Sri Bharati Krishna Tirtha Maharaj, who is generally considered the doyen of this discipline, in his seminal book Vedic Mathematics, wrote about this special use of verses in the Vedic age: "In order to help the pupil memorize the material assimilated, they made it a general rule of practice to write even the most technical and abstruse textbooks in sutras or in verse (which is so much easier - even for the children - to memorize)... So from this standpoint, they used verse for lightening the burden and facilitating the work (by versifying scientific and even mathematical material in a readily assimilable form)!"

Dr. L M Singhvi, the former High Commissioner of India in the UK, who is an avid endorser of the system says: "A single sutra would generally encompass a varied and wide range of particular applications and may be likened to a programmed chip of our computer age". Another Vedic maths enthusiast, Clive Middleton of vedicmaths.org feels, "These formulae describe the way the mind naturally works, and are therefore a great help in directing the student to the appropriate method of solution."

A Simple and Easy System

Practitioners of this striking method of mathematical problem-solving opine that Vedic maths is far more systematic, coherent and unified than the conventional system. It is a mental tool for calculation that encourages the development and use of intuition and innovation, while giving the student a lot of flexibility, fun and satisfaction. Therefore, it's direct and easy to implement in schools - a reason behind its enormous popularity among educationists and academicians.

Try These Out!

- If you want to find the square of 45, you can employ the Ekadhikena Purvena sutra ("By one more than the one before"). The rule says since the first digit is 4 and the second one is 5, you will first have to multiply 4 (4 +1), that is 4 × 5, which is equal to 20 and then multiply 5 with 5, which is 25. Viola! The answer is 2025. Now, you can employ this method to multiply all numbers ending with 5.
- If you want to subtract 4679 from 10000, you can easily apply the Nikhilam Navatashcaramam Dashatah sutra ("All from 9 and the last from 10"). Each figure in 4679 is subtracted from 9 and the last figure is subtracted from 10, yielding 5321. Similarly, other sutras lay down such simple rules of calculation.

> *"The world owes most to India in the realm of mathematics, which was developed in the Gupta period to a stage more advanced than that reached by any other nation of antiquity. The success of Indian mathematics was mainly due to the fact that Indians had a clear conception of the abstract number as distinct from the numerical quantity of objects or spatial extension." ~ A.L. Basham, Australian Indologist in The Wonder That Was India*

Born in the Vedic Age, but buried under centuries of debris, this remarkable system of calculation was deciphered towards the beginning of the 20th century, when there was a great interest in ancient Sanskrit texts, especially in Europe. However, certain texts called Ganita Sutras, which contained mathematical deductions, were ignored, because no one could find any mathematics in them. These texts, it's believed, bore the germs of what we now know as Vedic Mathematics.

Bharati Krishna Tirthaji's Discovery

Vedic math was rediscovered from the ancient Indian scriptures between 1911 and 1918 by Sri Bharati Krishna Tirthaji (1884-1960), a scholar of Sanskrit, Mathematics, History and Philosophy. He studied these ancient texts for years, and after careful investigation was able to reconstruct a series of mathematical formulae called sutras.

Bharati Krishna Tirthaji, who was also the former Shankaracharya (major religious leader) of Puri, India, delved into the ancient Vedic texts and established the techniques of this system in his pioneering work - Vedic Mathematics (1965), which is considered the starting point for all work on Vedic math. It is said that after Bharati Krishna's original 16 volumes of work expounding the Vedic system were lost, in his final years he wrote this single volume, which was published five years after his death.

Development of Vedic Math

Vedic math was immediately hailed as a new alternative system of mathematics, when a copy of the book reached London in the late 1960s. Some British mathematicians, including Kenneth Williams, Andrew Nicholas and Jeremy Pickles took interest in this new system. They extended the introductory material of Bharati Krishna's book, and delivered lectures on it in London. In 1981, this was collated into a book entitled Introductory Lectures on Vedic Mathematics. A few successive trips to India by Andrew Nicholas between 1981 and 1987, renewed the interest on Vedic math, and scholars and teachers in India started taking it seriously.

Interest in Vedic maths is growing in the field of education where maths teachers are looking for a new and better approach to the subject. Even students at IIT (Indian Institute of Technology) are said to be using this ancient technique for quick calculations. No wonder, a recent Convocation speech addressed to the students of IIT, Delhi, by Dr. Murli Manohar Joshi, Indian Minister for Science & Technology, stressed the significance of Vedic maths, while pointing out the important contributions of ancient Indian mathematicians, such as Aryabhatta, who laid the foundations of algebra, Baudhayan, the great geometer, and Medhatithi and Madhyatithi, the saint duo, who formulated the basic framework for numerals.

Vedic Maths in Schools

Quite a few years ago, St. James' School, London, and other schools began to teach the Vedic system, with notable success. Today this remarkable system is taught in many schools and institutes in India and abroad, and even to MBA and economics students.

When in 1988, Maharishi Mahesh Yogi brought to light the marvels of Vedic maths, Maharishi Schools around the world incorporated it in their syllabi. At the school in Skelmersdale, Lancashire, UK, a full course called "The Cosmic Computer" was written and tested on 11 to 14 year old pupils, and later published in 1998. According to Mahesh Yogi, "The sutras of Vedic Mathematics are the software for the cosmic computer that runs this universe."

Since 1999, a Delhi-based forum called International Research Foundation for Vedic Mathematics and Indian Heritage, which promotes value-based education, has been organising lectures on Vedic maths in various schools in Delhi, including Cambridge School, Amity International, DAV Public School, and Tagore International School.

Vedic Math Research

Researches are being undertaken in many areas, including the effects of learning Vedic maths on children. A great deal of research is also being done on how to develop more powerful and easy applications of the Vedic sutras in geometry, calculus, and computing. The Vedic Mathematics

Research Group published three new books in 1984, the year of the centenary of the birth of Sri Bharati Krishna Tirthaji.

Plus Points

There are obviously many advantages of using a flexible, refined and efficient mental system like Vedic math. Pupils can come out of the confinement of the 'only one correct' way, and make their own methods under the Vedic system. Thus, it can induce creativity in intelligent pupils, while helping slow-learners grasp the basic concepts of mathematics. A wider use of Vedic math can undoubtedly generate interest in a subject that is generally dreaded by children.

2

Swami Bharati Krishna Tirtha's Vedic Mathematics

Vedic mathematics is a system of mathematics consisting of a list of 16 basic sutras, or aphorisms, that allegedly encompass all mathematics. They were presented by a Hindu scholar and mathematician, Jagadguru Swami Sri Bharati Krishna Tirthaji Maharaja, during the early part of the 20th century (Trivedi, 1965).

Tirthaji claims that he found the sutras after years of studying the Vedas, a set of sacred ancient Hindu texts (Agrawala, 1992). However, labelling the mathematics he presented as 'Vedic' has provoked great controversy amongst Indian mathematicians who question both the Vedic origin of the mathematics, and whether the sutras can fulfil the claim of encompassing all mathematics. Nonetheless, the calculation strategies provided by Vedic mathematics are creative and useful, and can be applied in a number of ways to calculation methods in arithmetic and algebra, most notably within the education system. Vedic math has some similarities to the Trachtenberg system and many of the arithmetic computational strategies are based on the same concepts.

Tirthaji and the Discovery of Vedic Mathematics

Vedic mathematics was presented by Jagadguru Swami Sri Bharati Krsna Tirthaji, who is described as having the "rare combination of the probing insight and revealing intuition of a Yogi with the analytical acumen and synthetic talent of a mathematician" (Pratyagatmananda, 1965). Born in India in 1884, Tirthaji was an exceptional scholar; by age twenty he had studied at a number of colleges and universities throughout the country, been awarded the title of 'Saraswati' by the Madras Sanskrit Association for his remarkable proficiency in Sanskrit, and had completed seven masters degrees, including Sanskrit, Philosophy, English, Mathematics, History and Science, with the American College of Sciences (Trivedi, 1965).

Around 1911, Tirthaji resolved to study several sections of the Atharva-veda that had been dismissed by Orientalists, Indologists and antiquarian scholars as nonsensical (Tirthaji, 1992). He was part of a shrinking group of Indian scholars who believed that the Vedas represented an "inexhaustible mine of profound wisdom" both spiritual and secular (Pratyagatmananda, 1965). Tirthaji claimed that there were sections of the Atharva-veda labelled "ganita sutras" or "mathematical formulae" that mysteriously made no obvious reference to mathematics (Kansara, 1961). Tirthaji explains that he was determined to understand the "ganita sutra" references, and began studying ancient lexicons and lexicography in more detail (Kansara, 1961). With this resolve, Tirthaji went to Sringeri, Karnataka, where he began years of solitary study and meditation (Trivedi, 1965).

Eight years later, Tirthaji emerged claiming to have deciphered 16 fundamental mathematical sutras in the Vedas, which today have become the foundation of Vedic mathematics (Tirthaji, 1992). According to Tirthaji, the sutras cover every branch of mathematics, from arithmetic to spherical conics, and that "there is no mathematics beyond their jurisdiction" (Tirthaji, 1992).

After discovering the sutras, Tirthaji travelled around India presenting Vedic mathematics, and even lectured in the United States and England in 1958 (Trivedi, 1965). In addition to lecturing, Tirthaji also wrote sixteen volumes, one for each basic sutra, explaining their applications (Trivedi, 1965). Unfortunately, before they were published, the manuscripts were lost irretrievably (Kansara, 1961). Before falling ill and passing away in 1960, Tirthaji was able to rewrite the first of the sixteen volumes he had composed (Trivedi, 1965).

The Sutras

Vedic mathematics is based on sixteen sutras which serve as somewhat cryptic instructions for dealing with different mathematical problems.

Below is a list of the sutras, translated from Sanskrit into English:

- By one more than the one before
- "All from 9 and the last from 10"
- Vertically and crosswise
- Transpose and apply
- If the Samuccaya is the same it is zero
- If one is in ratio the other is zero
- By addition and by subtraction
- By the completion or non-completion
- Differential calculus
- By the deficiency
- Specific and general
- The remainders by the last digit
- The ultimate and twice the penultimate
- By one less than the one before
- The product of the sum
- All the multipliers

Subsutras or Corollaries

- "Proportionately"
- The remainder remains constant
- "The first by the first and the last by the last"
- For 7 the multiplicand is 143
- By osculation
- Lessen by the deficiency
- Whatever the deficiency lessen by that amount and set up the square of the deficiency
- Last Totalling 10
- Only the last terms
- The sum of the products
- By alternative elimination and retention
- By mere observation
- The product of the sum is the sum of the products
- On the flag

To check the factorization of polynomial expressions: "The product of the sum of the coefficients in the factors is equal to the sum of the coefficients in the product. The SC of the product (the expression) = Product of the SC (in the factors). Additionally, this sub-sutra helps to fill in the gaps when some factors are known.

Background Information on the Vedas

The word "veda" has two basic meanings. The first, a literal translation of the Sanskrit word, is "knowledge" (Veda). The second, and most common meaning of the word, refers to the sacred ancient literature of Hinduism, the Vedas, a collection of hymns, poetry and Hindu ceremonial formulas (Veda). Believed to be one of the oldest human written records, the Vedas date back over 4000 years (Gaskell, 2000). Traditionally, they were passed down orally and adapted from generation to generation by sacred sages called rishis, before eventually emerging written in Vedic, an ancient form of Sanskrit, during the 3rd century BCE (Veda).

The Vedas are divided into four main sections: the Rig-veda, Sama-veda, Yajurveda and the Atharva-veda, known collectively as the Samhitas (Veda). The first three, the Rig-veda, Sama-veda, and Yajurveda are basically ritual handbooks that were used by priests during the Vedic period (1500-500 BCE) (Veda). Vedic mathematics is apparently part of the fourth Veda, Atharva-veda, which is distinct from the others in several ways. First, unlike the religious focus of the other Vedas, the Atharva-veda contains hymns, spells and magical incantations for personal and domestic use (Veda). Also, the Atharva-veda, which was written later than the other Vedas, was not always considered authoritative, but only became so after being accepted by the Brahmans, the highest order of Hindu priests (Veda). Collectively, the Vedas do include information about a huge range of subjects, spanning religion, medicine, architecture, astronomy, etc. (Gaskell, 2000).

Although there is controversy about whether the Vedas themselves actually include references to mathematics, the roots of sophisticated mathematics have actually been traced back to the Vedic era. Ancient Indian Vedic civilisations are known for being skilled in geometry, algebra and computational mathematics complex enough to incorporate things like irrational numbers (Dutta, 2002). Furthermore, all ancient Indian mathematics literature is composed completely in verse; there was a tradition of composing terse sutras, like those of Vedic mathematics, to ensure that information would be preserved even if written records were damaged or lost (Dutta, 2002).

Applying the Sutras: Some of the Actual Mathematics

It is not difficult to understand and apply the Vedic mathematical strategies, as long as one does not rely on the sutras alone for mathematical insight. Those studying Vedic mathematics tend to strongly rely on the examples and explanations Tirthaji provides in his book.

All from Nine and the Last from Ten

When subtracting from a large power of ten with many columns of zeros you need not write the notation for "borrowing" from the column on the left. If you want to subtract 4679 from 10000, you can easily apply the Nikhilam Navatashcaramam Dashatah sutra ("[Subtract in] All [zero columns] from 9 and [in] the last [column subtract] from 10"). Each figure in 4679 is subtracted from 9 and the last figure is subtracted from 10, yielding 5321. This method is also used when finding the deficit from the next larger power of ten.

Corollary

"Whatever the extent of its deficiency, lessen it still further to that very extent; and also set up the square of that deficiency"

For instance, in computing the square of 9 we go through the following steps:

1. The nearest factor of 10 to 9 is 10. Therefore, let us take 10 as our base.
2. Since 9 is 1 less than 10, decrease it still further to 8. This is the left side of our answer.
3. On the right hand side put the square of the deficiency, which is 12. Hence the answer is 81.
4. Similarly, 82 = 64, 72 = 49.
    1. For numbers above 10, instead of looking at the deficit we look at the surplus. For example:

$11^2 = (11 + 1) \,.\, 10 + 1^2 = 121.$

$12^2 = (12 + 2) \,.\, 10 + 2^2 = 144.$

$14^2 = (14 + 4) \,.\, 10 + 4^2 = 18 \,.\, 10 + 16 = 196.$

$25^2 = [(25 + 5) . 2] . 10 + 5^2 = 625.$

$35^2 = [(35 + 5) . 3] . 10 + 5^2 = 40 . 3 . 10 + 25 = 1225.$

and so on.

This is based on the fact that a2 = (a + b)(a - b) + b2 where a is the number whose square is to be found and b is the deficit (or surplus) from nearest factor of 10.

By one More than the one Before

"Eka Dikenạ Parvesam" is the Sanskrit term for "One more than the previous one". It provides a simple way of calculating values like 1/x9 (e.g 1/19, 1/29, etc.). Let's take one 1/x9 and calculate, e.g., 1/19. In this case, x=1. The method is to write that x, then multiply it by 2 and keep that digit to its left. If the result of the multiplication is greater than 10, keep (value - 10) and keep the "1" as "carry over" which you'll add to the next digit.

The preposition "by" means the operations this formula concerns are either multiplication or division. [In case of addition/subtraction proposition "to" or "from" is used.] Thus this formula is used for either multiplication or division. It turns out that it is applicable in both operations.

Note: This applies to multiplication of numbers with the same first digit and the sum of their last unit digits is 10.

An interesting application of this formula is in computing squares of numbers ending in five. Consider:

$35 \times 35 = ((3 \times 3) + 3), 25 = 12, 25$

$125 \times 125 = ((12 \times 12) + 12), 25 = 156, 25$

The latter portion is multiplied by itself (5 by 5) and the previous portion is square of first digit or first two digit (3×3) or (12×12) and adding the same digit in that figure (3 or 12) resulting in the answer 1225.

This is a simple application of $(a + b)^2 = a^2 + 2ab + b^2$ when a = 10c and b = 5, i.e.

$$(10c + 6)^2 = 100c^2 + 100c + 25 = 100c\ (c + 1) = 25.$$

It can also be applied in multiplications when the last digit is not 5 but the sum of the last digits is the base (10) and the previous parts are the same. Consider:

$37 \times 33 = (3 \times 4),\ 7 \times 3 = 12,\ 21$

$29 \times 21 = (2 \times 3), 9 \times 1 = 6,\ 09$

This uses $(a + b)(a - b) = a^2 - b^2$ twice combined with the previous result to produce:

$$\begin{aligned}(10c + 5 + d)(10c + 5 - d) &= (10c + 5)^2 - d^2\\ &= 100c(c + 1) + 25 - d^2\\ &= 100c(c + 1) + (5 + d)(5 - d).\end{aligned}$$

We illustrate this formula by its application to conversion of fractions into their equivalent decimal form. Consider fraction 1/19. Using this formula, this can be converted into a decimal form in a single step. This can be done by applying the formula for either a multiplication or division operation, thus yielding two methods.

METHOD 1: USING MULTIPLICATION

1/19, since 19 is not divisible by 2 or 5, the fractional result is a purely circulating decimal. (If the denominator contains only factors 2 and 5 is a purely non-circulating decimal, else it is a mixture of the two.) Each factor of 2 or 5 or 10 in the denominator gives one fixed decimal digit.

So we start with the last digit

1

Multiply this by "one more", that is, 2 (this is the "key" digit from Ekadhikena)

21

Multiplying 2 by 2, followed by multiplying 4 by 2

421 → 8421

Now, multiplying 8 by 2, sixteen

68421

1 ← carry

multiplying 6 by 2 is 12 plus 1 carry gives 13

368421

1 ← carry

Continuing

7368421 → 47368421 → 947368421

1

Now we have 9 digits of the answer. There are a total of 18 digits (= denominator - numerator) in the answer computed by complementing the lower half (with its complement from nine):

052631578

947368421

Thus the result is 1/19 = 0.052631578,947368421 repeating.

1

21

421

8421

68421 (carry 1) - we got 16, so we keep 6 and carry 1

368421 (carry 1) - we get 6*2 + carry 1 = 13, so we keep 3 and carry one do this to eighteen digits (19-1. If you picked up 1/29, you'll have to do it till 28 digits). You'll get the following

1/19 = 052631578947368421

101001111010110000

Run this on your favourite calculator and check the result!

METHOD 2: USING DIVISION

The earlier process can also be done using division instead of multiplication. We divide 1 by 2, answer is 0 with remainder 1

.0

Next 10 divided by 2 is five

.05

Next 5 divided by 2 is 2 with remainder 1

.052

next 12 (remainder,2) divided by 2 is 6

.0526

and so on.

As another example, consider 1/7, this same as 7/49 which as last digit of the denominator as 9. The previous digit is 4, by one more is 5. So we muitiply (or divide) by 5, that is,

...7 ⇒ 57 ⇒ 857 ⇒ 2857 ⇒ 42857 ⇒ 142857 => .142,857 (stop after 7 - 1 digits)

3 2 4 1 2

Multiplying by 11

11*35= 385

(1) The five in the ones place of the answer is taken from the five in 35.

(2) The eight in the answer is the sum of 35 (3+5=8).

(3) The three in the hundreds place of the answer is taken from the three in 35.

However, if in step #2 the sum is greater than 9, the sum's left digit is added the first digit of the number multiplied by 11. For example:

11*59= 649

(1) The nine in the ones place of the answer is take from the nine in 59

(2) The four in the answer is the right digit in the sum of 59 $(5 + 9 = 14)$

(3) The six in the hundreds place of the answer is take from the sum of the five in 59 and the digit in the tens place from the sum of 59 $(5 + 9 = 14) \rightarrow (5 + 1 = 6)$

Vertically and Crosswise

This formula applies to all cases of multiplication and is very useful in division of one large number by another large number.

For example, to multiply 23 by 12:

$$\begin{array}{ccc} 2 & & 3 \\ | & \times & | \\ 1 & & 2 \\ 2\times1 & 2\times2+3\times1 & 3\times2 \\ 2 & 7 & 6 \end{array}$$

So $23 \times 12 = 276$.

When any of these calculations exceeds 9 then a carry is required. This is just $(10a+b)(10c+d)=100ac+10(ad+bc)+bd$.

Transpose and Apply

This formula complements "all from nine and the last from ten", which is useful in divisions by large numbers. This formula is useful in cases where the divisor consists of small digits. This formula can be used to derive the Horner's process of Synthetic Division.

When the Samuccaya is the Same, that Samuccaya is Zero

This formula is useful in solution of several special types of equations that can be solved visually. The word samuccaya has various meanings in different applications. For instance, it may mean a term which occurs as a common factor in all the terms concerned. A simple example is equation "$12x + 3x = 4x + 5x$". Since "x" occurs as a common factor in all the terms, therefore, $x = 0$ is a solution. Another meaning may be that samuccaya is a product of independent terms. For instance, in $(x + 7)(x + 9) = (x + 3)(x + 21)$, the samuccaya is $7 \times 9 = 3 \times 21$, therefore, $x = 0$ is a solution. Another meaning is the sum of the denominators of two fractions having the same numerical numerator, for example: $1/(2x - 1) + 1/(3x - 1) = 0$ means we may set the denominators equal to zero, $5x - 2 = 0$.

Yet another meaning is "combination" or total. This is commonly used. For instance, if the sum of the numerators and the sum of denominators are the same then that sum is zero. Therefore,

$$\frac{2x+9}{2x+7} = \frac{2x+7}{2x+9}$$

Therefore, $4x + 16 = 0$ or $x = -4$.

This meaning ("total") can also be applied in solving quadratic equations. The total meaning can not only imply sum but also subtraction. For instance when given $N_1/D_1 = N_2/D_2$, if $N_1 + N_2 = D_1 + D_2$ (as shown earlier) then this sum is zero. Mental cross multiplication reveals that the resulting equation is quadratic (the coefficients of x^2 are different on the two sides). So, if $N_1 - D_1 = N_2 - D_2$ then that samuccaya is also zero. This yields the other root of a quadratic equation.

Yet interpretation of "total" is applied in multi-term RHS and LHS. For instance, consider

$$\frac{1}{x-7} + \frac{1}{x-9} = \frac{1}{x-6} + \frac{1}{x-10}.$$

Here $D_1 + D_2 = D_3 + D_4 = 2x - 16$. Thus $x = 8$.

There are several other cases where samuccaya can be applied with great versatility. For instance "apparently cubic" or "biquadratic" equations can be easily solved as shown below:

$(x - 3)^3 + (x - 9)^3 = 2(x - 6)^3$.

Note that $x - 3 + x - 9 = 2(x - 6)$.

Therefore $(x - 6) = 0$ or $x = 6$.

This would not work for the apparently quadratic $(x - 3)^2 + (x - 9)^2 = 2(x - 6)^2$, which has no real or complex solutions.

Consider

$$\frac{(x+3)^3}{(x+5)^3} = \frac{x+1}{x+7}.$$

Observe: $N_1 + D_1 = N_2 + D_2 = 2x + 8$.

Therefore, $x = -4$.

This formula has been extended further.

If one is in Ratio, the other one is Zero

This formula is often used to solve simultaneous linear equations which may involve big numbers. But these equations in special cases can be visually solved because of a certain ratio between the coefficients. Consider the following example:

$$6x + 7y = 8$$
$$19x + 14y = 16$$

Here the ratio of coefficients of y is same as that of the constant terms. Therefore, the "other" variable is zero, i.e., $x = 0$. Hence, mentally, the solution of the equations is $x = 0$ and $y = 8/7$.

(alternatively:

$19x + 14y = 16$ is equivalent to:

$$(19/2)x + 7y = 8.$$

Thus it is obvious that x has to be zero, no ratio needed, just divide by 2!)

Note that it would not work if both had been "in ratio". For then we have the case of coinciding lines with an infinite number of solutions.:

$$6x + 7y = 8$$
$$12x + 14y = 16$$

This formula is easily applicable to more general cases with any number of variables. For instance

$$ax + by + cz = a$$
$$bx + cy + az = b$$
$$cx + ay + bz = c$$

which yields $x = 1$, $y = 0$, $z = 0$.

A corollary says solving "by addition and by subtraction." It is applicable in case of simultaneous linear equations where the x- and y-coefficients are interchanged. For instance:

$$45x - 23y = 113$$
$$23x - 45y = 91$$

By addition:

$$68x - 68y = 204$$
$$\Rightarrow \quad 68(x - y) = 204$$
$$\Rightarrow \quad x - y = 3.$$

By subtraction:

$$22x + 22y = 22$$
$$\Rightarrow \quad 22(x + y) = 22$$
$$\Rightarrow \quad x + y = 1.$$

Again, by addition, we eliminate the y-terms: $2x = 4$, so $x = 2$.

Or, by subtraction, we eliminate the x-terms: $-2y = 2$, and so $y = -1$.

The solution set is {2,-1}.

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Vedic Altars and the "Pythagorean Theorem"

A perfect example of this sort of misattribution involves the so-called Pythagorean theorem, the well-known theorem which was attributed to Pythagoras who lived around 500 BCE, but which was first proven in Greek sources in Euclid's Geometry, written centuries later. Despite the scarcity of evidence backing this attribution, it is not often questioned, perhaps due to the mantra—like frequency with which it is repeated. However, Seidenberg, in his 1978 article, shows that the thesis that Greece was the origin of geometric algebra was incorrect, "for geometric algebra existed in India before the classical period in Greece." (1978:323) It is now generally understood that the so-called "Pythagorean theorem" was understood in ancient India, and was in fact proved in Baudhayana's Shulva Sutra, a text dated to circa 600 BCE. (1978:323).

Knowledge of mathematics, and geometry in particular, was necessary for the precise construction of the complex Vedic altars, and mathematics was thus one of the topics covered in the brahmanas. This knowledge was further elaborated in the kalpa sutras, which gave more detailed instructions concerning Vedic ritual. Several of these treat the topic of altar construction. The oldest and most complete of these is the previously mentioned Shulva Sutra of

Baudhaayana. As this text was composed about a century before Pythagoras, the theory that the Greeks were the source of Geometric algebra is untenable, while the hypothesis that India was have been a source for Greek geometry, transmitted via the Persians who traded both with the Greeks and the Indians, looks increasingly plausible. On the other hand, it is quite possible that both the Greeks and the Indians developed geometry. Seidenberg has argued, in fact, that both seem to have developed geometry out of the practical problems involving their construction of elaborate sacrificial altars.

Zero and the Place Value System

Far more important to the development of modern mathematics than either Greek or Indian geometry was the development of the place value system of enumeration, the base ten system of calculation which uses nine numerals and zero to represent numbers ranging from the most minuscule decimal to the most inconceivably large power of ten. This system of enumeration was not developed by the Greeks, whose largest unit of enumeration was the myriad (10,000) or in China, where 10,000 was also the largest unit of enumeration until recent times. Nor was it developed by the Arabs, despite the fact that this numeral system is commonly called the Arabic numerals in Europe, where this system was first introduced by the Arabs in the thirteenth century.

Rather, this system was invented in India, where it evidently was of quite ancient origin. The Yajurveda Samhitaa, one of the Vedic texts predating Euclid and the Greek mathematicians by at least a millennium, lists names for each of the units of ten up to 10 to the twelfth power (paraardha). (Subbarayappa 1970:49) Later Buddhist and Jain authors extended this list as high as the fifty-third power, far exceeding their Greek contemporaries, who lacking a system of enumeration were unable to develop abstract mathematical concepts.

The place value system of enumeration is in fact built into the Sanskrit language, where each power of ten is given a distinct name. Not only are the units ten, hundred and thousand (daza, zata, sahasra) named as in English, but also ten thousand, hundred thousand, ten million, hundred million (ayuta, laksa, koti, vyarbuda), and so forth up to the fifty-third power, providing distinct names where English makes use of auxiliary bases such as thousand, million, etc. Ifrah has commented that:

> By giving each power of ten an individual name, the Sanskrit system gave no special importance to any number. Thus the Sanskrit system is obviously superior to that of the Arabs (for whom the thousand was the limit), or the Greeks and Chinese (whose limit was ten thousand) and even to our own system (where the names thousand, million etc. continue to act as auxiliary bases). Instead of naming the numbers in groups of three, four or eight orders of units, the Indians, from a very early date, expressed them taking the powers of ten and the names of the first nine units individually. In other words, to express a given number, one only had to place the name indicating the order of units between the name of the order of units immediately below it and the one immediately above it. That is exactly what is required in order to gain a precise idea of the place-value system, the rule being presented in a natural way and thus appearing self-explanatory. To put it plainly, the Sanskrit numeral system contained the very key to the discovery of the place-value system. (2000:429)

As Ifrah has shown at length, there is little doubt that our place-value numeral system developed in India (2000:399-409), and this system is embedded in the Sanskrit language, several aspects of which make it a very logical language, well suited to scientific and mathematical reasoning. Nor did this system exhaust Indian ingenuity; as van Nooten has shown, Pingala, who lived circa the first century BCE, developed a system of binary enumeration convertible to decimal numerals, described in his Chandahzaastra. His system is quite similar to that of Leibniz, who lived roughly fourteen hundred years later.

India is also the locus of another closely related an equally important mathematical discovery, the numeral zero. The oldest known text to use zero is a Jain text entitled the Lokavibhaaga, which has been definitely dated to Monday 25 August 458 CE. (Ifrah 2000:417-1 9) This concept, combined by the place-value system of enumeration, became the basis for a classical era renaissance in Indian mathematics.

The Indian numeral system and its place value, decimal system of enumeration came to the attention of the Arabs in the seventh or eighth century, and served as the basis for the well known advancement in Arab mathematics, represented by figures such as al-Khwarizmi. It reached Europe in the twelfth century when Adelard of Bath translated al-Khwarizmi's works into Latin. (Subbarayappa 1970:49) But the Europeans were at first resistant to this system, being attached to the far less logical roman numeral system, but their eventual adoption of this system led to the scientific revolution that began to sweep Europe beginning in the thirteenth century.

Luminaries of Classical Indian Mathematics

Aryabhatta

The world did not have to wait for the Europeans to awake from their long intellectual slumber to see the development of advanced mathematical techniques. India achieved its own scientific renaissance of sorts during its classical era, beginning roughly one thousand years before the European Renaissance. Probably the most celebrated Indian mathematicians belonging to this period was Aryabhatta, who was born in 476 CE.

In 499, when he was only 23 years old, Aryabhatta wrote his Aaryabhattiya, a text covering both astronomy and mathematics. With regard to the former, the text is notable for its for its awareness of the relativity of motion. This awareness led to the astonishing suggestion that it is the Earth that rotates the Sun. He argued for the diurnal rotation of the earth, as an alternate theory to the rotation of the fixed stars and sun around the earth (Pingree 1981:18). He made this suggestion approximately one thousand years before Copernicus, evidently independently, reached the same conclusion.

With regard to mathematics, one of Aryabhatta's greatest contributions was the calculation of sine tables, which no doubt was of great use for his astronomical calculations. In developing a way to calculate the sine of curves, rather than the cruder method of calculating chords devised by the Greeks, he thus went beyond geometry and contributed to the development of trigonometry, a development which did not occur in Europe until roughly one thousand years later, when the Europeans translated Indian influenced Arab mathematical texts.

Aryabhatta's mathematics was far ranging, as the topics he covered include geometry, algebra, trigonometry. He also developed methods of solving quadratic and indeterminate equations using fractions. He calculated pi to four decimal places, i.e., 3.1416. (Pingree 1981:57) In addition, Aryabhatta "invented a unique method of recording numbers which required perfect understanding of zero and the place-value system." (Ifrah 2000:419)

Given the astounding range of advanced mathematical concepts and techniques covered in this fifth century text, it should be of no surprise that it became extremely well known in India, judging by the large numbers of commentaries written upon it. It was studied by the Arabs in the eighth century following their conquest of Sind, and translated into Arabic, whence it influenced the development of both Arabic and European mathematical traditions.

Brahmagupt

Born in 598 CE in Rajasthan in Western India, Brahmagupta founded an influential school of mathematics which rivalled Aryabhatta's. His best known work is the Brahmasphuta Siddhanta, written in 628 CE, in which he developed a solution for a certain type of second order indeterminate equation.

Mahavira

Mahavira was a Jain mathematician who lived in the ninth century, who wrote on a wide range of mathematical topics. These include the mathematics of zero, squares, cubes, square-roots, cube-roots, and the series extending beyond these. He also wrote on plane and solid geometry, as well as problems relating to the casting of shadows. (Pingree 1981:60)

Bhaskar

Bhaskar was one of the many outstanding mathematicians hailing from South India. Born in 1114 CE in Karnataka, he composed a four-part text entitled the Siddhanta Ziromani. Included in this compilation is the Biijaganita, which became the standard algebra textbook in Sanskrit. It contains descriptions of advanced mathematical techniques involving both positive and negative integers as well as zero, irrational numbers. It treats at length the "pulverizer" (kuttakaara) method of solving indeterminate equations with continued fractions, as well as the so-called "Pell's equation (vargaprakrti) dealing with indeterminate equations of the second degree. He also wrote on the solution to numerous kinds of linear and quadratic equations, including those involving multiple unknowns, and equations involving the product of different unknowns. (Pingree 1981, p. 64)

In short, he wrote a highly sophisticated mathematical text that proceeded by several centuries the development of such techniques in Europe, although it would be better to term this a rediscovery, since much of the Renaissance advances of mathematics in Europe was based upon the discovery of Arab mathematical texts, which were in turn highly influenced by these Indian traditions.

Madhava

The Kerala region of South India was home to a very important school of mathematics. The best known member of this school Madhava (c. 1444-1545), who lived in Sangamagrama in Kerala. Primarily an astronomer, he made history in mathematics with his writings on trigonometry. He calculated the sine, cosine and arctangent of the circle, developing the world's first consistent system of trigonometry. He also correctly calculated the value of p to eleven decimal places.

This is by no means a complete list of influential Indian mathematicians or Indian contributions to mathematics, but rather a survey of the highlights of what is, judged by any fair, unbiased standard, an illustrious tradition, important both for its own internal elegance as well as its influence on the history of European mathematical traditions. The classical Indian mathematical renaissance was an important precursor to the European renaissance, and to ignore this fact is to fail to grasp the history of latter, a history which was truly multicultural, deriving its inspiration from a variety of cultural roots.

There are in fact, as Frits Staal has suggested in his important (1995) article, "The Sanskrit of Science", profound similarities between the social contexts of classical India and renaissance Europe. In both cases, important revolutions in scientific thought occurred in complex, hierarchical societies in which certain elite groups were granted freedom from manual labour, and thus the opportunity to dedicate themselves to intellectual pursuits. In the case of classical India, these groups included certain brahmins as well as the Buddhist and Jain monks, while in renaissance Europe they included both the monks as well as their secular derivatives, the university scholars.

Why, one might ask, did Europe take over thousand years to attain the level of abstract mathematics achieved by Indians such as Aryabhatta? The answer appears to be that Europeans were trapped in the relatively simplistic and concrete geometrical mathematics developed by the Greeks. It was not until they had, via the Arabs, received, assimilated and accepted the place-value system of enumeration developed in India that they were able to free their minds from the concrete and develop more abstract systems of thought. This development thus triggered the scientific and information technology revolutions which swept Europe and, later, the world. The role played by India in the development is no mere footnote, easily and inconsequentially swept under the rug of Eurocentric bias. To do so is to distort history, and to deny India one of it's greatest contributions to world civilization.

Jagadguru Swami Sri Bharati Krishna Tirthaji Maharaja

Jagadguru Swami Sri Bharati Krsna Tirthaji was the Jagadguru (literally, teacher of the world; assigned to heads of Hindu mathas) of the Govardhana Matha of Puri during 1925-1960. He was one of the most significant spiritual figures in Hinduism during the 20th century. He is particularly known for his work on Vedic mathematics.

Early Life: Venkatraman Shastri was born in March, 1884 to P. Narasimha Shastri, originally a tehsildar at Tirunelveli in Madras Presidency. Narasimha Shastri later became the Deputy Collector of the Presidency. Venkatraman was born in a highly illustrious family. His uncle, Chandrasekhara Shastri was the Principal of the Maharaja's College in Vizianagaram, while his great-grandfather, Justice C. Ranganath Shastri was a judge in the Madras High Court.

Educational Career: Venkatraman Shastri started his educational career as a student of the National College in Trichanapalli. After that he moved to the Church Missionary Society College and eventually the Hindu College, both in Tirunelveli. He was consistently first place in all subjects in all of his classes. Shastri passed his matriculation examination from the Madras University in January, 1899, where he also finished at the head of the class.

As a student Venkatraman was marked for his splendid brilliance, superb retentive memory and an insatiable curiosity. By deluging his teachers with piercing questions, making them uneasy, and frequently forcing them to admit ignorance he was considered a terribly mischievous student.

Although Venkatraman always scored high in subjects like mathematics, sciences and humanities, he was also proficient in languages and particularly adept in Sanskrit. According to his own testimonials, Sanskrit and oratory were his favourite subjects. Such was his mastery over the language, that he was awarded the title "Saraswati" by the Madras Sanskrit Association in July, 1899 at the age of 16. At about that time, Venkatraman was profoundly influenced by his Sanskrit guru Sri Vedam Venkatrai Shastri whom he remembered with deepest love, reverence and gratitude, with tears in his eyes.

Venkatraman won the highest place in the graduation B.A. examination in 1902. He then appeared for the M.A. Examination for the American College of Sciences, in Rochester, New York from the Bombay centre in 1903. He passed the M.A. examination in seven subjects that he had chosen — Sanskrit, philosophy, English, mathematics, history, science and another — simultaneously scoring the highest honours in all, which was perhaps an all-time world record at the time.

Venkatraman Saraswati, as he was called after receiving the title, also contributed to William Thomas Stead's Review of Reviews on topics as diverse as religion and science. During his college days, he also wrote extensively on history, sociology, philosophy, politics and literature. Reading of the latest scientific research and discoveries was his hobby throughout his life.

Early Public Life: Venkatraman Saraswati worked under Gopal Krishna Gokhale in 1905 for the National Education Movement and the South African Indian problems. However, his inclination towards science and Indic studies led him to study the ancient Indian holy scriptures, Adhyatma-Vidya. In 1908 he joined the Sringeri Matha in Mysore to study under the Sringeri Shankaracharya Sri Satchidananda Sivabhinava Nrisimha Bharati Swami. However, his spiritual practise was interrupted when he was pressurized by nationalist leaders to head the newly started National College at Rajmahendri. Prof. Venkatraman Saraswati taught at the college for three years. But in 1911, he suddenly left the college to go back to Sri Satchidananda Sivabhinava Nrisimha Bharati Swami at the Sringeri Math in his quest for spiritual knowledge.

Spiritual Path: Returning to Sringeri, Venkatraman spent his next eight years studying advanced Vedanta philosophy at the feet of Shri Nrisimha Bharati Swami. He also practised vigorous meditation, Brahma-sadhana and Yoga-sadhana during those years in the nearby forests. It is believed that he attained spiritual self-realization during his years in the Sringeri Math. He would leave the material world and practise Yoga meditation in seclusion for many days. During those eight years, he also taught Sanskrit and Philosophy to local schools and ashrams. He delivered a series of sixteen lectures on Shankaracharya's philosophy at

Shankar Institute of Philosophy, Amalner [Khandesh]. During that time, he also lectured as a guest professor at various institutions in Bombay, Poona and Khandesh.

Initiation into Sanyasa Order: After Venkatraman's eight-year period of spiritual practice and study of the Vedanta and Vedic philosophy, he was initiated into the holy order of Samnyasa at Benaras by Jagadguru Shankaracharya Sri Trivikram Tirthaji Maharaj of Sharadapeeth on July 4, 1919 and on this occasion he was given the title of Swami and the new name, "Swami Bharati Krsna Tirtha".

Shankaracharya of Sharada Peetha: Swami Bharati Krsna Tirtha was installed as Shankaracharya of Sharada Peetha in 1921 after just two years of Sanyasa. After assuming the pontificate Shri Jagadguruji, he was given another title, Jagadguru, as is the tradition. The Swami then toured India from corner to corner giving lectures on Sanatana Dharma, Vedic philosophy and Vedanta. By his scintillating intellectual brilliance, powerful oratory, magnetic personality, sincerity of purpose, indomitable will, purity of thought, and loftiness of character he took the entire intellectual and religious class by storm.

Shankaracharya of Govardhan Math: Around the time the Swami became Shankaracharya of Sharada Peetha, the Shankaracharya of Govardhan Math Puri, Jagadguru Shankaracarya Sri Madhusudhan Tirtha, was in failing health and was greatly impressed by Bharati Krishna Tirtha. Madhusudan Tirtha requested Bharati to succeed him at the Govardhan Math, however the Swami respectfully declined the offer. However, in 1925, Shankaracarya Sri Madhusudhan Tirtha's health took a serious turn and Swami Bharati Krsna Tirtha had to accept the Govardhan Math's Gadi. In 1925, Swami Bharati Krsna Tirtha assumed the pontificate of Shankaracharya of Govardhan Math, Puri and relinquished the pontificate of Sharadapeeth Gadi of Sringeri. He installed Sri Swarupanandji as the new Shankaracharya of Sharada Peetha.

Jagadguru: After becoming the Shankaracharya of Govardhan Math, Swami Bharati toured all over the world for 35 years to spread the values of peace, harmony and brotherhood and to spread the message of the Sanatana Dharma. He took upon himself the colossal task of the renaissance of Indian culture.

He attended many national and international religious conferences and many other yoga workshops. He believed in the Vedantic ideal of "Purnatva" which literally translated means, "all-round perfection and harmony". He remained the Shankaracharya of the Govardhan Matha until his death in 1960. While being a pontiff, he wrote a large number of treatises and books on religion, sciences, mathematics, world peace and social issues. In 1953, at Nagpur, he founded an organization called "Sri Vishwa Punarnirmana · Sangha" (World Reconstruction Association). Initially, the administrative board consisted of Jagadguruji's disciples, devotees and admirers of his spiritual ideals for humanitarian service, but later many distinguished people started to contribute to the mission. The Chief Justice of India, Justice B. P. Sinha served as its President. Dr. C. D. Deshmukh, the ex-Finance Minister of India and ex-Chairman or the University Grants Commission served as its Vice-President.

In February 1958 he went on a trans-oceanic tour to America to speak on world peace and Vedanta, staying three months in Los Angeles, California travelling via the United Kingdom. This was the first tour outside India by a Shankaracharya in the history of the order. The tour was sponsored by Self Realization Fellowship of Los Angeles, the Vedantic Society founded by Paramhansa Yogananda in America.

In 1965 a Chair of Vedic Studies was founded at Banares Hindu University by Shri Arvind N. Mafatlala, a generous Bombay business magnate and math devotee of the late Swami Shankaracarya.

Mathematics: Jagadguru Swami Sri Bharati Krsna Tirthaji Maharaja's book "Vedic Mathematics" opened the floodgates of similar literature, often derived from the Swami's 16 sutras themselves. His treatise on this field of mathematics is a fundamental work on speed and accuracy in basic mathematics. The Vedic Math ideal is a mental calculation and one-line notation.

The foundations of Vedic Mathematics were mentioned in the Vedas themselves and even in the Vedanta scriptures. These had lain unused for many millennia, till the Swami rediscovered them.

His book, Vedic Mathematics, comprises many algorithms. He revealed his source in the ancient Hindu Vedas. Some are intuitively reconstructed from the Atharva Veda and from Parisistas (appendix) of the Atharva Veda. "The Upaveda of Sthapatya (engineering) comprises all kinds of architectural and structural human endeavour and all visual arts (and mathematics)." His work seems to be a whole Parisistas (appendix) itself.

The ancient Sanskrit writers did not use numerals when writing big numbers but preferred to use the letters of the Sanskrit Devanagari alphabet. In the Vedic Sutras the key word steps to solving many problems are given in a terse, decimal code of certain sets of rhyming syllables, within the verses of the Sutra.

The fact that the alphabetic code is in the natural order and can be immediately interpreted, is clear proof that the code language was resorted not for concealment but for greater ease in verification.

The Swami had written sixteen volumes on the Vedic Mathematics field explaining all the topics of mathematical study. Alas, many advanced formula were promised but not given in his first and only book. After his 1956 life's work manuscript on Vedic mathematics was lost in a fire at the home of a disciple, though he was going blind from cataracts, he re-wrote the manuscript in 1957 in six weeks! It was to be proofread and published in the USA but was send back to India in 1960 after his death. In 1965, this manuscript was published by Motilal Banarsidass, Varanasi, India and reprinted four times in the 1970s.

His book, Vedic Mathematics, included sixteen terse formulas for mental mathematics. For arithmetic, we are given several algorithms for whole number multiplication and division, (flag or straight) division, fraction conversion to repeating decimal numbers, calculations with measures of mixed units, summation of a series, squares and square roots (duplex method), cubes and cube roots (with expressions for a digit schedule), and divisibility (by osculation).

He gives a poem in Anusub metre, couched in the alphabetic Code-Language that has three meanings, a hymn to Lord Sri Krsna, a hymn in praise of the Lord Shri Shankara, and the third the value of pi/10 to 32 decimal places, pi/10 = 0.31415926535897932384626433832792... with a "self-contained master-key" for extending the evaluation to any number of decimal places!

Several tests and techniques for factoring and solving certain algebraic equations with integer roots for quadratic, cubic, biquadratic, pentic equations, systems of linear equations, and systems of quadratic equations are demonstrated. For fractional expressions, a separation algorithm and fraction merger algorithms are given. Other techniques handle certain patterns of some special case algebraic equations. Just an introduction to differential and integral calculus is given.

Geometric applications are reviewed for linear equations, analytic conics, the equation for the asymptotes, and the equation to the conjugate-hyperbola. Five simple geometric proofs for the Pythagorean theorem are given. A 5-line proof of Apollonius' Theorem is given.

Advanced topics promised included the integral calculus (the centre of gravity of hemispheres, conics), Trigonometry, Astronomy (spherical triangles, earth's daily rotation, earth's annual rotation about the sun and eclipses), and Engineering (dynamics, statics, hydrostatics, pneumatics, applied mechanics).

In his final comments he asserted that the names for "Arabic numerals," "Pythagoras' Theorem," and "Cartesian" coordinates are historical misnomers.

Advanced topics presented included the integral calculus, the centre of gravity of hemisphere, cone, etc. Trigonometry, Astronomy (spherical triangles, earth's daily rotation, as one manual says about the sun and the moon), and Engineering (dynamics, statics, hydrostatics, pneumatics, applied mechanics).

In his final comments he asserted that the names for "Arabic numerals," "Pythagoras' Theorem," and "Cartesian" coordinates are historical misnomers.

4

Basic Principles

The basic principles involved are quite simple; it is the algorithmic aspects that require attention in particular cases.

Assume a rational function R(x) in one indeterminate x has denominator that factors as

P(x)Q(x)

over a field K (we can take this to be real numbers, or complex numbers). If P and Q have no common factor, then R may be written as

A/P + B/Q

for some polynomials A(x) and B(x) over K. The existence of such a decomposition is a consequence of the fact that the polynomial ring over K is a principal ideal domain, so that

CP + DQ = 1

for some polynomials C(x) and D(x).

Using this idea inductively we can write R(x) as a sum with denominators powers of irreducible polynomials. To take this further, if required, write

$G(x)/F(x)^n$

as a sum with denominators powers of F and numerators of degree less than F, plus a possible extra polynomial. This can be done by the Euclidean algorithm, polynomial case.

Therefore when K is the complex numbers and we can assume F has degree 1 (by the fundamental theorem of algebra) the numerators will be constant. When K is the real numbers we can have the case of

degree F = 2,

and a quotient of a linear polynomial by a power of a quadratic will occur. This therefore is a case that requires discussion, in the systematic theory of integration (for example in computer algebra).

Algorithms

Lagrange Interpolation

Partial fraction decomposition can be derived using Lagrange interpolation.

EXAMPLE

As an introductory example we take the rational function

$$\frac{x}{x^2-1}.$$

This by the difference of two squares identity can also be written as

$$\frac{x}{(x+1)(x-1)},$$

which can be transformed further. Consider an identity

$$\frac{A}{x+1}+\frac{B}{x-1}=\frac{x}{x^2-1},$$

where A and B are constants. In more explicit form, we have the relation of the numerators,

$$A(x - 1) + B(x + 1) = x.$$

We know that the constants on one side of an expression must equal those on the other side. On the left hand side, the constants are -A and B, and on the right, the constant is simply 0. So, comparing constants on both sides of the expression, we can see that

$$B - A = 0,$$

i.e. $A = B$.

Now, in the same way, we know that the number of x terms on the left must equal the number of x's on the right. Therefore, looking at x terms on both sides,

$$Ax + Bx = x,$$

therefore

$$A + B = 1,$$

and so, given that $A = B$, we can say that

$$A + A = 1$$

$$2A = 1$$

$$A = B = 0.5$$

Finally we find:

$$\frac{x}{x^2-1}=\frac{1}{2}.\frac{1}{x+1}+\frac{1}{2}.\frac{1}{x-1}$$

or

$$\frac{x}{x^2-1}=\frac{1}{2(x+1)}+\frac{1}{2(x-1)}$$

which holds true for all $x \neq \pm1$.

Derivation

The preceding example can be generalized to the following situation:

Assume that $Q(x)$ is a monic polynomial of some degree n which over the underlying field K decomposes into linear factors

$$Q(x)=\prod_{i=1}^{n}(x-x_i),$$

where all xi are pairwise different. In other words Q has simple roots (over K). If P(x) is any polynomial of degree $\le n - 1$ then according to the Lagrange interpolation formula P(x) can be uniquely written as a sum (the Lagrange form representation)

$$P(x) = \sum_{j=1}^{n} P(x_j) L_j(x; x_j),$$

where L_j (x; x_j) is the Lagrange polynomial

$$L_j(x; x_j) = \prod_{k \le n,\, k \ne j} \frac{(x - x_k)}{(x_j - x_k)}.$$

Dividing the Lagrange representation on the right side termwise by the polynomial Q(x) in its factored form one obtains

$$\frac{P(x)}{Q(x)} = \sum_{j=1}^{n} \frac{P(x_j)}{\prod k \le n,\, k \ne j(x_j - x_k)} . \frac{1}{x - x_j}.$$

This is the partial fraction decomposition

$$\frac{P(x)}{Q(x)} = \sum_{j=1}^{n} \frac{P(x_j)}{Q'(x_j)} . \frac{1}{x - x_j}$$

of the rational function R(x) = P(x)/Q(x) with the derivative of Q given by

$$Q'(x_j) = \sum_{l \le n} \prod_{k \le n,\, k \ne l} (x_j - x_k) = \prod_{k \le n,\, k \ne j} (x_j - x_k)$$

The first example can be obtained as the special case

Q(x) = (x - 1) . (x + 1), P(x) = x

Note the close relationship to divided differences.

Paravartya Sutra

Separation of a fractional algebraic expression into partial fractions is the reverse of the process of combining fractions by converting each fraction to the lowest common denominator (LCM) and adding the numerators. This separation is accomplished by a mental, one-line Vedic formula called the Paravartya Sutra. Case one has fractional expressions where factors in the denominator are unique. Case two has fractional expressions where some factors may repeat as powers of a binomial.

In integral calculus we would want to write a fractional algebraic expression as the sum of its partial fractions in order to take the integral of each simple fraction separately. Once the original denominator, D0, has been factored we set up a fraction for each factor in the denominator. We may use a subscripted D to represent the denominator of the respective partial fractions which are the factors in D0. Letters A, B, C, D, E, and so on will represent the numerators of the respective partial fractions.

We calculate each respective numerator by (1) calculating the Paravartya value of the denominator (which is the value of the variable making that factor zero) and (2) then substituting this value into the original expression but ignoring that factor in the denominator. Each Paravartya value for the variable is the value which gives an undefined value to the expression since we do not divide by zero.

General formula:

$$\frac{lx^2+mx+n}{(x-a)(x-b)(x-c)}=\frac{A}{(x-a)}+\frac{B}{(x-b)}+\frac{C}{(x-c)}$$

Here, a, b, c, l, m, and n are given integer values.

Where x = a and

$$A=\frac{la^2+ma+n}{(a-b)(a-c)};$$

and where x = b and

$$B=\frac{lb^2+mb+n}{(b-c)(b-a)};$$

and where x = c and

$$C = \frac{lc^2 + mc + n}{(c-a)(c-b)};$$

Case One

Factorize the expression in the denominator. Set up a partial fraction for each factor in the denominator. Apply the Paravartya Sutra to solve for the new numerator of each partial fraction.

EXAMPLE

$$\frac{3x^2 + 12x + 11}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

Set up a partial fraction for each factor in the denominator. With this framework we apply the sutra to solve for A, B, and C by mental math.

1. D_1 is x + 1; set it equal to zero. This gives the Paravartya value for A when x = -1.
2. Next, substitute this value of x into the fractional expression, but without D_1.

3A. Put this value down as the value of A. Proceed similarly for B and C.

3B. For Paravartya B use x = -2.

3C. For Paravartya C use x = -3.

Thus, to solve for A, use x = -1:

$$\frac{3x^2 + 12x + 11}{(x+2)(x+3)} = \frac{3 - 12 + 11}{(1)(2)} = \frac{2}{2} = 1 = A.$$

Thus, to solve for B, use x = -2:

$$\frac{3x^2 + 12x + 11}{(x+2)(x+3)} = \frac{12 - 24 + 11}{(-1)(1)} = \frac{-1}{(-1)} = +1 = B.$$

Thus, to solve for C, use x = -3:

$$\frac{3x^2 + 12x + 11}{(x+2)(x+3)} = \frac{27 - 36 + 11}{(-2)(-1)} = \frac{2}{(+2)} = +1 = C.$$

Thus,

$$\frac{3x^2+12x+11}{(x+1)(x+2)(x+3)}=\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}$$

Case Two

When factors of the denominator include powers of one expression we (1) Set up a partial fraction for each unique factor and each lower power of D; (2) We set up an equation showing the relation of the numerators if all were converted to the LCD. From the equation of numerators we solve for each numerator, A, B, C, D, and so on. This equation of the numerators is an absolute identity, true for all values of x. So, we may select any value of x and solve.

EXAMPLE

$$\frac{3x+5}{(1-2x)^2}=\frac{A}{(1-2x)^2}+\frac{B}{(1-2x)}$$

Here, we set up a partial fraction for each descending power of the denominator. Then we solve for the numerators, A and B. As the Paravartya value for A and B will be the same, x = ½, we need an additional relation in order to solve for both. To write the relation of numerators the second fraction needs another factor of (1-2x) to convert it to the LCD, giving us 3x + 5 = A + B(1 - 2x).

To Solve for A: Set the denominator of the first fraction to zero, 1 - 2x = 0. This is the Paravartya value for A, x = ½. When we substitute this value, x = ½, into the relation of numerators we have 3(1/2) + 5 = A + B(0). Solving for A gives us A = 3/2 + 5 = 13/2. Numerator A equals six and one-half.

To Solve for B: Since the equation of the numerators, 3x + 5 = A + B(1 - 2x), is true for all values of x, pick a value for x and use it to solve for B. We have solved for the value of A above, A = 13/2. We may use it to solve for B.

We may pick x = 0, use A = 13/2, and solve for B.

$$3x + 5 = A + B(1 - 2x)$$
$$0 + 5 = 13/2 + B(1 + 0)$$
$$10/2 = 13/2 + B$$
$$-3/2 = B$$

We may pick x = 1.

Then solve for B:

$$3x + 5 = A + B(1 - 2x)$$
$$3 + 5 = 13/2 + B(1 - 2)$$
$$8 = 13/2 + B(-1)$$
$$16/2 = 13/2 - B$$
$$B = -3/2$$

We may pick x = -1. Solve for B:

$$3x + 5 = A + B(1 - 2x)$$
$$-3 + 5 = 13/2 + B(1 + 2)$$
$$4/2 = 13/2 + 3B$$
$$-9/2 = 3B$$
$$-3/2 = B$$

Hence,

$$\frac{3x+5}{(1-2x)^2} = \frac{13/2}{(1-2x)^2} + \frac{-3/2}{(1-2x)},$$

or

$$\frac{3x+5}{(1-2x)^2} = \frac{13}{2(1-2x)^2} + \frac{3}{2(1-2x)}.$$

Technique Three

A third technique is an analysis of the expanded relation of the numerators. Just match up the x-terms of each degree. Then one can see the coefficients of the matching terms and solve for the missing numerator.

EXAMPLE

$$\frac{3x+5}{(1-2x)^2}=\frac{A}{(1-2x)^2}+\frac{B}{(1-2x)}$$

Converting each fraction to the LCD we have the relation in the numerators: $3x + 5 = A + B(1 - 2x)$. As A and B are constants, we can expand and match up the constant terms and the x-terms. The values of A and B will then be apparent.

$$3x + 5 = A + B(1 - 2x)$$

$$3x + 5 = A + B - 2Bx$$

Hence, the constant terms are set equal and the x-terms are set equal:

$$5 = A + B$$

$$3x = -2Bx$$

Therefore, by setting the coefficients equal,

$$3 = -2B$$

$$-3/2 = B$$

And,

$$5 = A - 3/2$$

$$5 + 3/2 = A$$

$$13/2 = A$$

Hence,

$$\frac{3x+5}{(1-2x)^2}=\frac{13/2}{(1-2x)^2}+\frac{-3/2}{(1-2x)}.$$

Fractions of Integers

The idea of partial fractions can be generalized to other rings, say the ring of integers where prime numbers take the role of irreducible denominators. E.g., it is:

$$\frac{1}{18}=\frac{1}{2}-\frac{1}{3}-\frac{1}{3^2}.$$

Partial Fraction Decomposition over the Reals

In mathematics, partial fractions are used in real-variable integral calculus to find real-valued antiderivatives of rational functions. The partial fraction decomposition of real rational functions is also used for Laplace transforms.

This article describes the general method for obtaining the partial fraction decomposition of any real rational function. The proof of the existence of the partial fraction decomposition over an arbitrary field is not given here. For a sketch of the general proof, see partial fraction.

For applications of partial fraction decomposition over the reals, see

- Partial fractions in integration
- Partial fractions in Laplace transforms

First we state the general result, and then we offer several examples to illustrate the result and show how the method is used in practice.

General Result

Let f(x) be any rational function over the real numbers. In other words, suppose there exist real polynomials p(x) and $q(x) \neq 0$, such that

$$f(x) = \frac{p(x)}{q(x)}$$

By removing the leading coefficient of q(x), we may assume without loss of generality that q(x) is monic. By the fundamental theorem of algebra, we can write

$$q(x) = (x - a_1)^{j_1} \ldots (x - a_m)^{j_m} \left(x^2 + b_1 x + c_1\right)^{k_1} \ldots \left(x^2 + b_n x + c_n\right)^{k_n}$$

where $a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_n$ are real numbers with $b_i^2 - 4c_i < 0$, and $j_1, \ldots, j_m, k_1, \ldots, k_n$ are positive integers. The terms $(x - a_i)$ are the linear factors of q(x) which correspond to real roots of q(x), and the terms $(x_i^2 + b_i x + c_i)$ are the irreducible quadratic factors of q(x) which correspond to pairs of complex conjugate roots of q(x).

Then the partial fraction decomposition of f(x) is the following.

$$f(x)=\frac{p(x)}{q(x)}=P(x)+\sum_{i=1}^{m}\sum_{r=1}^{j_i}\frac{A_{ir}}{(x-a_i)^r}+\sum_{i=1}^{n}\sum_{r=1}^{k_i}\frac{B_{ir}x+c_{ir}}{(x^2+b_ix+c_i)^r}$$

Here, P(x) is a (possibly zero) polynomial, and the Air, Bir, and Cir are real constants. There are a number of ways the constants can be found.

The most straightforward method is to multiply through by the common denominator q(x). We then obtain an equation of polynomials whose left-hand side is simply p(x) and whose right-hand side has coefficients which are linear expressions of the constants A_{ir}, B_{ir}, and C_{ir}. Since two polynomials are equal if and only if their corresponding coefficients are equal, we can equate the coefficients of like terms. In this way, a system of linear equations is obtained which always has a unique solution. This solution can be found using any of the standard methods of linear algebra. However, this is often not the best way to go when computing by pencil and paper, and there are other ways ("tricks") to obtain the constants.

The Algorithm

The partial fraction decomposition of a real rational function can be obtained by the following procedure:

1. Remove the leading coefficient of the denominator q(x) if necessary.
2. Reduce the rational function as much as possible; i.e. cancel any obvious common factors of the numerator and denominator.
3. Make sure the degree of the numerator p(x) is strictly smaller than the degree of the denominator q(x). If it is not, then use the division algorithm for

polynomials and long-divide the denominator into the numerator. You will get a (nonzero) polynomial quotient and a remainder whose degree is now strictly smaller than the degree of the denominator. Take this remainder as your new numerator and continue with the following steps. (Don't forget the polynomial quotient when you're done!)

4. Factor the denominator q(x) into linear and irreducible quadratic factors. This is not a straightforward task; however, you have some tools to use. If you find any real root a of q(x), then you know that (x - a) is a factor of q(x). Also, if you differentiate q(x) and use the Euclidean algorithm to find the greatest common divisor of q' (x) and q(x), then any real root a of the greatest common divisor gives you a factor (x - a). You can look for ways to complete the square in different powers of x. Most importantly, you can use the rational root test to determine all possible rational roots of q(x).
5. Express the partial fraction decomposition of f(x) as the sum of double sums above.
6. Find the constants A_{ir}, B_{ir}, and C_{ir} by whatever methods possible. (These methods are described below.) Once you have found all the constants, you now have the partial fraction decomposition. (Don't forget the polynomial quotient!)

Some Examples

EXAMPLE 1

$$f(x) = \frac{1}{x^2 + 2x - 3}$$

Here, the denominator splits into two distinct linear factors:

$q(x) = x^2 + 2x - 3 = (x + 3)(x - 1)$

so we have the partial fraction decomposition

$$f(x)=\frac{1}{x^2+2x-3}=\frac{A}{x+3}+\frac{B}{x-1}$$

Multiplying through by $x^2 + 2x - 3$, we have the polynomial identity

$$1 = A(x - 1) + B(x + 3)$$

Substituting $x = -3$ and $x = 1$ into this equation gives $A = -1/4$ and $B = 1/4$, so that

$$f(x)=\frac{1}{x^2+2x-3}=\frac{1}{4}\left(\frac{-1}{x+3}+\frac{1}{x-1}\right)$$

EXAMPLE 2

$$f(x)=\frac{x^3+16}{x^3-4x^2+8x}$$

After long-division, we have

$$f(x)=1+\frac{4x^2-8x+16}{x^3-4x^2+8x}=1+\frac{4x^2-8x+16}{x\left(x^2-4x+8\right)}$$

Since $(-4)^2 - 4(8) = -16 < 0$, $x^2 - 4x + 8$ is irreducible, and so

$$\frac{4x^2-8x+16}{x(x^2-4x+8)}=\frac{A}{x}+\frac{Bx+C}{x^2-4x+8}$$

Multiplying through by x3 - $4x^2 + 8x$, we have the polynomial identity

$$4x^2 - 8x + 16 = A(x^2 - 4x + 8) + (Bx + C)x$$

Taking $x = 0$, we see that $16 = 8A$, so $A = 2$. Comparing the x^2 coefficients, we see that $4 = A + B = 2 + B$, so $B = 2$. Comparing linear coefficients, we see that $-8 = -4A + C = -8 + C$, so $C = 0$. Altogether,

$$f(x)=1+2\left(\frac{1}{x}+\frac{x}{x^2-4x+8}\right)$$

The following example illustrates almost all the "tricks" one would need to use short of consulting a computer algebra system.

EXAMPLE 3

$$f(x)=\frac{x^9-2x^6+2x^5-7x^4+13x^3-11x^2+12x-4}{x^7-3x^6+5x^5-7x^4+7x^3-5x^2+3x-1}$$

After long-division and factoring, we have

$$f(x)=x^2+3x+4+\frac{x^5-2x^4+5x^3-5x^2+6x-1}{(x-1)^3(x^2+1)^2}$$

The partial fraction decomposition takes the form

$$\frac{x^5-2x^4+5x^3-5x^2+6x-1}{(x-1)^3(x^2+1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{Dx+E}{x^2+1}+\frac{Fx+G}{(x^2+1)^2}$$

Multiplying through by (x - 1)3(x^2 + 1)^2 we have the polynomial identity

$x^5 - 2x^4 + 5x^3 - 5x^2 + 6x - 1$

$= A(x - 1)^2 (x^2 + 1)^2 + B(x - 1)$

$(x^2 + 1)^2 + C(x^2 + 1)^2 + (Dx + E)$

$(x - 1)^3 (x^2 + 1) + (Fx + G)(x - 1)^3$

Taking x = 1 gives 4 = 4C, so C = 1. Similarly, taking x = i gives 2 + 2i = (Fi + G)(2 + 2i), so Fi + G = 1, so F = 0 and G = 1 by equating real and imaginary parts.

We now have the identity

$x^5 - 2x^4 + 5x^3 - 5x^2 + 6x - 1$

$= A(x-1)^2(x^2 + 1)^2 + B(x - 1)(x^2 + 1)^2$

$+ (x^2+1)^2 + (Dx+E)(x - 1)^3(x^2 + 1)$

$+ (x - 1)^3$

Taking constant terms gives E = A - B + 1, taking leading coefficients gives A = -D, and taking x-coefficients gives B = 3 - D - 3E.

Putting all of this together, E = A - B + 1 = -D - (3 - D - 3E) + 1 = 3E - 2, so E = 1 and A = B = -D.

Now,

$x^5 - 2x^4 + 5x^3 - 5x^2 + 6x - 1$

$= A(x-1)^2(x^2 + 1)^2 + A(x - 1)(x^2 + 1)^2$

$+ (x^2 +1)^2 + (-Ax + 1)(x-1)^3(x^2+ 1) + (x - 1)^3$

Taking $x = -1$ gives $-20 = -8A - 20$, so $A = B = D = 0$. The partial fraction decomposition of f(x) is thus

$$f(x) = x^2 + 3x + 4 + \frac{1}{(x-1)^3} + \frac{1}{x^2+1} + \frac{1}{(x^2+1)^2}$$

General Methods for Finding Coefficients

We present in the sequel a list of methods for finding the coefficients occurring in the r.h.s. sums of the above equation for f(x)=p(x)/q(x)=..., to which we shall refer to by (PFD).

For some of the methods it would be useful if p and/or q were already written in factored form.

Simple Poles of Multiplicity 1

If ai is a simple pole of f, i.e. $j_i=1$, then one multiplies (PFD) by $(x-a_i)$ and takes the limit x→ai. On the r.h.s. only A_i1 "survives", which is therefore given by

$A_i1 = \lim_{x\to a}i\ (x-a_i)\ f(x) = p(a_i)/q_i(a_i)$

where q_i is q divided (simplified) by the factor $(x - a_i)$. It holds $q_i(a_i) = q'(a_i)$.

Leading Term of a Pole

Using the same idea, we determine the coefficients A_{ij_i} (multiplying in the highest negative power), by just replacing $(x-a_i)$ by its j_i-th power.

Leading Term of a Quadratic Factor

Still using the same idea, but going to the complex root of $x^2+b\ x+c$, one gets in the same way coefficients B_{ik_i}, C_{ik_i}

(from real and imaginary part of the previous equation for A which is now an equation for Bx + C).

Other Coefficients A

All coefficients relative to one pole can be obtained "in one step" by making a change of variables $t = x - a_i$ (i.e. replacing x by $t + a_i$) and then (long) dividing the "new" p by the "new" q, with the "pure" power t_{ji} removed.

Method of Derivatives

Instead of making the change of variables and using long division, one can obtain a recursive formula involving derivatives of p and q_i at $x = a_i$.

General Methods

Besides the above methods to obtain a particular coefficient, there are several general methods to get one or more equations involving one or several of the coefficients, which should then allow to determine the remaining unknown coefficients:

- method of particular values: this just consists in putting x equal to particular fixed values (0, ±1, ±2...), and thus obtaining numerical equations for the coefficients.
- method of limits: multiply both sides of (PFD), after subtracting known terms (especially the polynomial), by the highest possible power of x such that $\lim(x \neq 8)$ still is finite, and take this limit
- method of identification: the "final rescue", put all of the r.h.s. on the common denominator q(x) and identify coefficients of alike powers in the numerator on the r.h.s. and p(x) on the l.h.s.

Partial Fractions in Complex Analysis

In complex analysis, a partial fraction expansion is a way of writing a meromorphic function f(z) as an infinite sum of rational functions and polynomials. When f(z) is a rational function, this reduces to the usual method of partial fractions.

Motivation

By using polynomial long division and the partial fraction technique from algebra, any rational function can be written as a sum of terms of the form 1/(az + b)k + p(z), where a and b are complex, k is an integer, and p(z) is a polynomial. Just as polynomial factorization can be generalized to the Weierstrass factorization theorem, there is an analogy to partial fraction expansions for certain meromorphic functions.

A proper rational function, i.e. one for which the degree of the denominator is greater than the degree of the numerator, has a partial fraction expansion with no polynomial terms. Similarly, a meromorphic function f(z) for which |f(z)| goes to 0 as z goes to infinity at least as quickly as |1/z|, has an expansion with no polynomial terms.

Calculation

Let f(z) be a function meromorphic in the finite complex plane with poles at $\lambda_1, \lambda_2, \ldots$, and let $(\Gamma_1, \Gamma_2, \ldots)$ be a sequence of simple closed curves such that:

- The origin lies inside each curve Gk
- No curve passes through a pole of f
- Γk lies inside Γ_{k+1} for all k

  $\lim\limits_{k\to\infty} d(\Gamma_k) = \infty$, where d(Gk) gives the distance from the curve to the origin

Suppose also that there exists an integer p such that

$$\lim_{k\to\infty} \oint \Gamma_k \left| \frac{f(z)}{z^{p+1}} \right| dz < \infty$$

Writing PP(f(z); z = λ_k) for the principal part of the Laurent expansion of f about the point λ_k, we have

$$f(z)=\sum_{k=0}^{\infty}PP\big(f(z);z=\lambda_k\big)$$

if p = -1, and if p > -1,

$$f(z)=\sum_{k=0}^{\infty}PP\big(f(z);z=\lambda_k+c_{0,k}+c_{1,k}z+\ldots+c_{p,k}z^p\big)$$

where the coefficients $c_{j,k}$ are given by

$$c_{j,k}=\operatorname{Res}_{z=\lambda_k}\frac{f(z)}{z^{j+1}}$$

λ_0 should be set to 0, because even if f(z) itself does not have a pole at 0, the residues of f(z)/zj+1 at z = 0 must still be included in the sum.

Note that in the case of $\lambda_0 = 0$, we can use the Laurent expansion of f(z) about the origin to get

$$f(z)=\frac{a_{-m}}{z^m}+\frac{a_{-m+1}}{z^{m-1}}+\ldots+a_0+a_1z+\ldots$$

$$c_{j,k}=\operatorname{Res}_{z=0}\left(\frac{a_{-m}}{z^{m+j+1}}+\frac{a_{-m+1}}{z^{m+j}}+\ldots+\frac{a_j}{z}+\ldots\right)=a_j,$$

$$\sum_{j=0}^{p}c_{j,k}z^j=a_0+a_1z+\ldots+a_pz^p$$

so that the polynomial terms contributed are exactly the regular part of the Laurent series up to z^p.

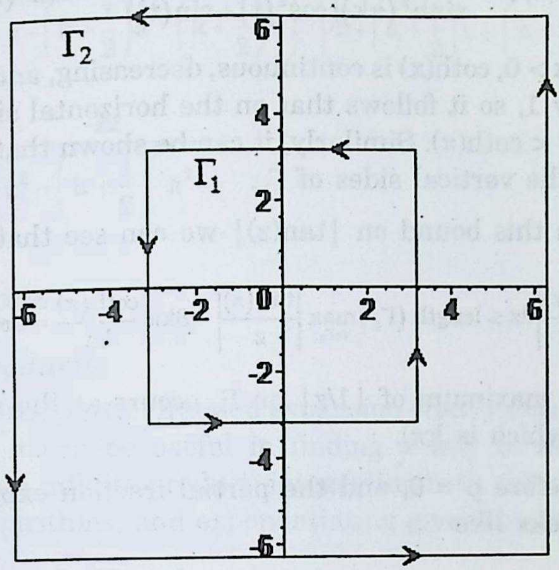
For the other poles λ_k where k = 1, $1/z^{j+1}$ can be pulled out of the residue calculations:

$$c_{j,k}=\frac{1}{\lambda_k^{j+1}}\operatorname{Res}_{z=\lambda_k}f(z)$$

$$\sum_{j=0}^{p}c_{j,k}z^j=\left[\operatorname{Res}_{z=\lambda_k}f(z)\right]\sum_{j=0}^{p}\frac{1}{\lambda_k^{j+1}}z^j$$

To avoid issues with convergence, the poles should be ordered so that if λ_k is inside Γ_n, then λ_j is also inside Γ_n for all j < k.

EXAMPLE



The simplest examples of meromorphic functions with an infinite number of poles are the non-entire trigonometric functions, so take the function tan(z). tan(z) is meromorphic with poles at $(n + 1/2)\pi$, $n = 0, \pm1, \pm2, \ldots$ The contours Γ_k will be squares with vertices at $\pm\pi k \pm \pi k i$ traversed counterclockwise, $k > 1$, which are easily seen to satisfy the necessary conditions.

On the horizontal sides of Γ_k,

$z = t \pm \pi k i,\ t \in [-\pi k, \pi k]$,

so

$$|\tan(z)|^2 = \left|\frac{\sin(t)\cosh(\pi k) \pm i\cos(t)\sinh(\pi k)}{\cos(t)\cosh(\pi k) \pm i\sin(t)\sinh(\pi k)}\right|^2$$

$$|\tan(z)|^2 = \frac{\sin^2(t)\cosh^2(\pi k) \pm \cos^2(t)\sinh^2(\pi k)}{\cos^2(t)\cosh^2(\pi k) \pm \sin^2(t)\sinh^2(\pi k)}$$

sinh(x) < cosh(x) for all real x, which yields

$$|\tan(z)|^2 < \frac{\cosh^2(\pi k)\left(\sin^2(t)+\cos^2(t)\right)}{\sinh^2(\pi k)\left(\cos^2(t)+\sin(t)\right)} = \coth^2(\pi k)$$

For $x > 0$, coth(x) is continuous, decreasing, and bounded below by 1, so it follows that on the horizontal sides of Γ_k, $|\tan(z)| < \coth(\pi)$. Similarly, it can be shown that $|\tan(z)| < 1$ on the vertical sides of Γ_k.

With this bound on $|\tan(z)|$ we can see that

$$\oint\Gamma_k \left|\frac{\tan(z)}{z}\right| dz \le \text{length}\,(\Gamma_k)\max_{z\in\Gamma_k}\left|\frac{\tan(z)}{z}\right| < 8k\pi\,\frac{\coth(\pi)}{k\pi} = 8\coth(\pi) < \infty.$$

(The maximum of $|1/z|$ on Γ_k occurs at the minimum of $|z|$, which is $k\pi$).

Therefore $p = 0$, and the partial fraction expansion of tan(z) looks like

$$\tan(z) = \sum_{k=0}^{\infty}\left(PP\left(\tan(z); z=\lambda_k\right) + \text{Res}_{z=\lambda_k}\frac{\tan(z)}{z}\right)$$

The principal parts and residues are easy enough to calculate, as all the poles of tan(z) are simple and have residue -1:

$$PP\left(\tan(z); z=\left(n+\frac{1}{2}\right)\pi\right) = \frac{-1}{z-\left(n+\frac{1}{2}\right)\pi}$$

$$\text{Res}_{z=\left(n+\frac{1}{2}\right)\pi}\frac{\tan(z)}{z} = \frac{-1}{\left(n+\frac{1}{2}\right)\pi}$$

We can ignore $\lambda_0 = 0$, since both tan(z) and tan(z)/z are analytic at 0, so there is no contribution to the sum, and ordering the poles λ_k so that $\lambda_1 = \pi/2$, $\lambda_2 = -\pi/2$, $\lambda_3 = 3\pi/2$, etc., gives

$$\tan(z)=\sum_{k=0}^{\infty}\left[\left(\frac{-1}{z-\left(k+\frac{1}{2}\right)\pi}-\frac{1}{\left(k+\frac{1}{2}\right)\pi}\right)+\left(\frac{-1}{z+\left(k+\frac{1}{2}\right)\pi}+\frac{1}{\left(k+\frac{1}{2}\right)\pi}\right)\right]$$

$$\tan(z)=\sum_{k=0}^{\infty}\frac{-2z}{z^2-\left(k+\frac{1}{2}\right)^2\pi^2}$$

Applications

Infinite Products

Because the partial fraction expansion often yields sums of 1/(a+bz), it can be useful in finding a way to write a function as an infinite product; integrating both sides gives a sum of logarithms, and exponentiating gives the desired product:

$$\tan(z)=\sum_{k=0}^{\infty}\left(\frac{1}{z-\left(k+\frac{1}{2}\right)\pi}+\frac{1}{z+\left(k+\frac{1}{2}\right)\pi}\right)$$

$$\int_0^z\tan(w)\,dw=\log\ \sec z$$

$$\int_0^z\frac{1}{w\pm\left(k+\frac{1}{2}\right)\pi}\,dw=\log\left(1\pm\frac{z}{\left(k+\frac{1}{2}\right)\pi}\right)$$

Applying some logarithm rules,

$$\log\sec z=-\sum_{k=0}^{\infty}\left(\log\left(1-\frac{z}{\left(k+\frac{1}{2}\right)\pi}\right)+\log\left(1+\frac{z}{\left(k+\frac{1}{2}\right)\pi}\right)\right)$$

$$\log\sec z = -\sum_{k=0}^{\infty}\log\left(1-\frac{z^2}{\left(k+\frac{1}{2}\right)^2\pi^2}\right),$$

which finally gives

$$\cos z = \prod_{k=0}^{\infty}\left(1-\frac{z^2}{\left(k+\frac{1}{2}\right)^2\pi^2}\right)$$

Laurent Series

The partial fraction expansion for a function can also be used to find a Laurent series for it by simply replacing the rational functions in the sum with their Laurent series, which are often not difficult to write in closed form. This can also lead to interesting identities if a Laurent series is already known.

Recall that

$$\tan(z) = \sum_{k=0}^{\infty}\frac{-2z}{z^2-\left(k+\frac{1}{2}\right)^2\pi^2} = \sum_{k=0}^{\infty}\frac{-8z}{4z^2-(2k+1)^2\pi^2}$$

We can expand the summand using a geometric series:

$$\frac{-8z}{4z^2-(2k+1)^2\pi^2} = \frac{8z}{(2k+1)^2\pi^2}\cdot\frac{1}{1-\left(\frac{2z}{(2k+1)\pi}\right)^2}$$

$$= \frac{8}{(2k+1)^2\pi^2}\sum_{n=0}^{\infty}\frac{2^{2n}}{(2k+1)^{2n}\pi^{2n}}z^{2n+1}$$

Substituting back,

$$\tan(z) = 2\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{2^{2n+2}}{(2k+1)^{2n+2}\pi^{2n+n}}z^{2n+1},$$

which shows that the coefficients an in the Laurent (Taylor) series of tan(z) about z = 0 are

$$a_{2n+1} = \frac{T_{2n+1}}{(2n+1)!} = \frac{2^{2n+3}}{\pi^{2n+2}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+2}}$$

$$a_{2n} = \frac{T_{2n}}{(2n)!} = 0,$$

where T_n are the tangent numbers.

Conversely, we can compare this formula to the Taylor expansion for tan(z) about z = 0 to calculate the infinite sums:

$$\tan(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \ldots$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{2^3} = \frac{\pi^2}{8}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{1}{3}\frac{\pi^4}{2^5} = \frac{\pi^4}{96}$$

Line Integral

In mathematics, a line integral (sometimes called a path integral) is an integral where the function to be integrated is evaluated along a curve. Various different line integrals are in use. In the case of a closed curve it is also called a contour integral.

The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve (commonly arc length or, for a vector field, the scalar product of the vector field with a differential vector in the curve). This weighting distinguishes the line integral from simpler integrals defined on intervals. Many simple formulas in physics (for example, $W = \vec{F}.\vec{d}$) have natural continuous analogs in terms of line integrals $\left(W = \int_C \vec{F}.d\vec{s}\right)$. The line integral finds the work done on an object moving through an electric or gravitational field, for example.

Vector Calculus

In qualitative terms, a line integral in vector calculus can be thought of as a measure of the total effect of a given vector field along a given curve.

DEFINITION

For some scalar field $f : U \rightarrow R$ defined on an open subset U of R^n, the line integral on a curve C, parametrized as r(t).U with $t \in [a, b]$ is defined by

$$\int_C f\ ds = \int_a^b f(r(t))\,|\,r'(t)\,|\,dt.$$

where

f is the scalar field being integrated,

r(t): [a, b] → C is a bijective parametrization of the curve C and

r(a) and r(b) give the endpoints of C.

The symbol ds is heuristically interpreted as an elementary arc length. Because they depend only on the element of arc length, line integrals of scalar fields are independent of the parametrization r(t).

For a vector field $F : U \,.\, R^n \rightarrow R^n$, the line integral on a curve C, parametrized as r(t).U with t . [a, b], is defined by

$$\int_C F(r).dr = \int_a^b F(r(t)).r'(t)dt.$$

Line integrals of vector fields are independent of the parametrization r(t) in absolute value, but they do depend on its orientation. Specifically, a reversal in the orientation of the parametrization changes the sign of the line integral.

Path Independence

If a vector field F is the gradient of a scalar field G, that is,

$$\nabla G = F,$$

then the derivative of the composition of G and r(t) is

$$\frac{dG(r(t))}{dt} = \nabla G(r(t)).r'(t) = F(r(t)).r'(t)$$

which happens to be the integrand for the line integral of F on r(t). It follows that, given a path C, then

$$\int_C F(r).dr = \int_a^b F\left(r(t)\right).r'(t)dt$$

$$= \int_a^b \frac{dG\left(r(t)\right)}{dt}\,dt = G\left(r(b)\right) - G\left(r(a)\right).$$

In words, the integral of F over C depends solely on the values of G in the points r(b) and r(a) and is thus independent of the path between them.

For this reason, a line integral of a vector field which is the gradient of a scalar field is called path independent.

Applications

The line integral has many uses in physics. For example, the work done on a particle travelling on a curve C inside a force field represented as a vector field F is the line integral of F on C.

Relationship with the Line Integral in Complex Analysis

Viewing complex numbers as 2D vectors, the line integral in 2D of a vector field corresponds to the real part of the line integral of the conjugate of the corresponding complex function of a complex variable.

Due to the Cauchy-Riemann equations the curl of the vector field corresponding to the conjugate of a holomorphic function is zero. This relates through Stokes theorem both types of line integral being zero.

Complex Analysis

The line integral is a fundamental tool in complex analysis. Suppose U is an open subset of C, $\gamma : [a, b] \to U$ is a rectifiable curve and $f : U \to C$ is a function. Then the line integral

$$\int_\gamma f(z)dz$$

may be defined by subdividing the interval [a, b] into $a = t_0 < t_1 < \ldots < t_n = b$ and considering the expression

$$\sum_{1 \le k \le n} f(\gamma(t_k))(\gamma(t_k) - \gamma(t_{k-1})).$$

The integral is then the limit of this sum, as the lengths of the subdivision intervals approach zero.

If γ is a continuously differentiable curve, the line integral can be evaluated as an integral of a function of a real variable:

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

When γ is a closed curve, that is, its initial and final points coincide, the notation

$$\oint_\gamma f(z)dz$$

is often used for the line integral of f along γ.

Important statements about contour integrals are the Cauchy integral theorem and Cauchy's integral formula.

Because of the residue theorem, one can often use contour integrals in the complex plane to find integrals of real-valued functions of a real variable (see residue theorem for an example).

EXAMPLE

Consider the function f(z)=1/z, and let the contour C be the unit circle about 0, which can be parametrized by e^{it}, with t in $[0,2\pi]$. Substituting, we find

$$\oint_C f(z)dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}dt = i\int_0^{2\pi} e^{-it}e^{it}dt$$

$$= i\int_0^{2\pi} dt = i(2\pi - 0) = 2\pi i$$

which can be also verified by the Cauchy integral formula.

Quantum Mechanics

The "path integral formulation" of quantum mechanics actually refers not to path integrals in this sense but to functional integrals, that is, integrals over a space of paths, of a function of a possible path. However, path integrals in the sense of this article are important in quantum mechanics; for example, complex contour integration is often used in evaluating probability amplitudes in quantum scattering theory.

5

Methods of Contour Integration

In complex analysis, contour integration is a method of evaluating certain integrals of real-valued functions along intervals on the real line that are not readily found by using only real variables. Complex analysis methods described below outline the procedures for calculating these real-valued integrals by means of contour integrals in the complex plane.

These methods include

- direct integration of a complex-valued function along a curve in the complex plane (a contour)
- application of the Cauchy integral formula
- application of the residue theorem

One method can be used, or a combination of these methods, or various limiting processes, for the purpose of finding these integrals or sums.

Direct Methods

Direct methods involve the calculation of the integral by means of methods similar to those in calculating line integrals in several-variable calculus. This means that we use the following method:

- parametrizing the contour

The contour is parametrized by a differentiable complex-valued function of real variables, or the contour is broken up into pieces and parametrized separately

- substitution of the parametrization into the integrand

Substituting the parametrization into the integrand transforms the integral into an integral of one real variable.

- direct evaluation

The integral is evaluated in a method akin to a real-variable integral.

EXAMPLE

A fundamental result in complex analysis is that the integral around the contour C which is the unit circle (or any Jordan curve about 0) of z^{-1} is $2\pi i$. Let us evaluate the integral

$$\oint_C \frac{1}{z} dz.$$

In evaluating this integral, we use the unit circle $|z| = 1$ as our contour, which we can parametrize by $\gamma(t) = e^{it}$, with t . $[0, 2\pi]$. Observe that $\pi'(t)$ = ieit. Now, substituting this for z, we have

$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} dt$$

$$= i\int_0^{2\pi} e^{-it}e^{it} dt = i\int_0^{2\pi} 1\,dt$$

$$= [t]_0^{2\pi}\ i = (2\pi - 0)i = 2\pi i$$

which is the value of the integral.

Applications of Integral Theorems

Applications of integral theorems are also often used to evaluate the contour integral along a contour, which means that the real-valued integral is calculated simultaneously along with calculating the contour integral.

Integral theorems such as the Cauchy integral formula or residue theorem are generally used in the following method:

- a specific contour is chosen.

The contour is chosen so that the contour follows the part of the complex plane that describes the real-valued integral, and also encloses singularities of the integrand so application of the Cauchy integral formula or residue theorem is possible

- application of the Cauchy-Goursat theorem

The integral is reduced to only an integration around a small circle about each pole.

- application of the Cauchy integral formula or residue theorem

Application of these integral formula gives us a value for the integral around the whole of the contour.

- division of the contour into a contour along the real part and imaginary part

The whole of the contour can be divided into the contour that follows the part of the complex plane that describes the real-valued integral as chosen before (call it R), and the integral that crosses the complex plane (call it I). The integral over the whole of the contour is the sum of the integral over each of these contours.

- demonstration that the integral that crosses the complex plane plays no part in the sum

If the integral I can be shown to be zero, or if the real-valued integral that is sought is improper, then if we demonstrate that the integral I as described above tends to 0, the integral along R will tend to the integral around the contour R+I.

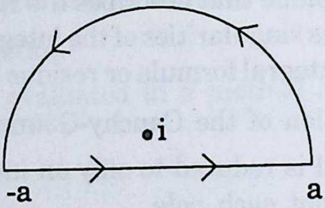
- conclusion

If we can show the above step, then we can directly calculate R, the real-valued integral.

EXAMPLE (I)

Consider

$$\int_{-\infty}^{\infty}\frac{1}{\left(x^2+1\right)^2}dx$$



To evaluate this integral, we look at the complex-valued function

$$f(z)=\frac{1}{\left(z^2+1\right)^2}$$

which has singularities at i and -i. However, we will want to choose a contour that will enclose the real-valued integral, so we choose a semicircle as the one to the left, which we will let expand as to contain the whole real axis (a will tend to infinity). Call this contour C.

Now, there are two ways of proceeding, using the Cauchy integral formula or by the method of residues.

Using the Cauchy Integral Formula

Observe that

$$f(z)=\frac{1}{\left(z^2+1\right)^2}=\frac{1}{(z+i)^2(z-i)^2}.$$

Since the only singularity in the contour is the one at i, then we can write

$$f(z)=\frac{\frac{1}{(z+i)^2}}{(z-i)^2},$$

which puts the function in the form for direct application of the formula.

By the formula, then,

$$\oint_C f(z)\,dz = \oint_C \frac{1}{\left(z^2+1\right)^2}\,dz.$$

$$= \oint_C \frac{\frac{1}{(z+i)^2}}{(z-i)^2}\,dz$$

$$= 2\pi i \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)\Bigg|_{z=i}$$

$$= 2\pi i\left(\frac{-2}{(z+i)^3}\right)\Bigg|_{z=i}$$

$$= 2\pi i(-i/4) = \frac{\pi}{2}$$

If we call the arc of the semicircle A, we need to show that the integral over A tends to zero as a tends to infinity-using the estimation lemma

$$\left|\int_A f(z)\,dz\right| \le ML$$

where M is an upper bound on $|f(z)|$ and L the length of A. Now,

$$\int_A f(z)\,dz \le \frac{a\pi}{\left(a^2+1\right)^2} \to 0 \text{ as } a \to \infty$$

So

$$\int_{-\infty}^{\infty} f(z)\,dz = \frac{\pi}{2}.$$

Using the Method of Residues

Consider the Laurent series of f(z) about i, the only singularity we need to consider. We then have

$$f(z)=\frac{1}{4(z-i)^2}+\frac{-i}{4(z-i)}+\frac{3}{16}+\frac{i}{8}(z-i)+\frac{-5}{64}(z-i)^2+\ldots$$

It is clear by inspection that the residue is -i/4, so, by the residue theorem, we have

$$\oint_C f(z)dz=\oint_C \frac{1}{\left(z^2+1\right)^2}dz=2\pi i \operatorname{Res}_{z=i} f=2\pi i(-i/4)=\frac{\pi}{2}$$

If we call the arc of the semicircle A, we need to show that the integral over A tends to zero as a tends to infinity - using the estimation lemma

$$\left|\oint_C f(z)dz\right|\le ML$$

where M is an upper bound on $|f(z)|$ and L the length of A. Now,

$$\oint_C f(z)dz\le\frac{a\pi}{\left(a^2+1\right)^2}\to 0 \text{ as } a\to\infty.$$

So

$$\int_{\infty}^{\infty} f(z)dz=\frac{\pi}{2}.$$

Thus we get the same result as before.

Contour Note

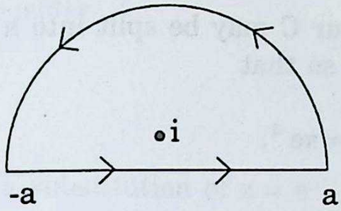
As an aside, a question can arise whether we do not take the semicircle to include the other singularity, enclosing -i. To have the integral along the real axis moving in the correct direction, the contour must travel clockwise, i.e., in a negative direction, reversing the sign of the integral overall.

This does not affect the use of the method of residues by series.

EXAMPLE (II) - CAUCHY DISTRIBUTION

The integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx$$



(which arises in probability theory as (a scalar multiple of) the characteristic function of the Cauchy distribution) resists the techniques of elementary calculus. We will evaluate it by expressing it as a limit of contour integrals along the contour C that goes along the real line from -a to a and then counterclockwise along a semicircle centred at 0 from a to -a. Take a to be greater than 1, so that the imaginary unit i is enclosed within the curve. The contour integral is

$$\int_C \frac{e^{itz}}{z^2+1}\,dz.$$

Since e^{itz} is an entire function (having no singularities at any point in the complex plane), this function has singularities only where the denominator $z^2 + 1$ is zero. Since $z^2 + 1 = (z + i)(z - i)$, that happens only where $z = i$ or $z = -i$. Only one of those points is in the region bounded by this contour. The residue of f(z) at z = i is

$$\lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{e^{itz}}{z^2+1}$$

$$= \lim_{z\to i}(z-i)\frac{e^{itz}}{(z-i)(z+i)}$$

$$= \lim_{z \to i} (z - i) \frac{e^{iti}}{i + i} = \frac{e^{-t}}{2i}.$$

According to the residue theorem, then, we have

$$\int_C f(z)dz = 2\pi i. \operatorname{Res}_{z=i} f(z) = 2\pi i \frac{e^{-t}}{2i} = \pi e^{-t}.$$

The contour C may be split into a "straight" part and a curved arc, so that

$$\int_{\text{straight}} + \int_{\text{arc}} = \pi e^{-1},$$

and thus

$$\int_{-a}^{a} = \pi e^{-t} - \int_{\text{arc}}.$$

It can be shown that if t > 0 then

$$\int_{\text{arc}} \frac{e^{itz}}{z^2 + 1} dz \to 0 \text{ as } a \to \infty$$

Therefore if t > 0 then

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2 + 1} dz = \pi e^{-t}.$$

A similar argument with an arc that winds around -i rather than i shows that if t < 0 then

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2 + 1} dz = \pi e^{t},$$

and finally we have

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2 + 1} dz = \pi e^{-|t|}.$$

(If t = 0 then the integral yields immediately to real-valued calculus methods and its value is p.)

EXAMPLE (III) - TRIGONOMETRIC INTEGRALS

Certain substitutions can be made to integrals involving trigonometric functions, so the integral is transformed into a rational function of a complex variable and then the above methods can be used in order to evaluate the integral.

As an example, consider

$$\int_{-\pi}^{\pi} \frac{1}{1+3(\cos t)^2}\,dt.$$

We seek to make a substitution of $z = e^{it}$.

Now, recall

$$\cos t = \frac{1}{2}\left(e^{it} + e^{-it}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and

$$\frac{dz}{dt} = iz,\ dt = \frac{dz}{iz}.$$

Taking C to be the unit circle, we substitute to get:

$$\oint_C \frac{1}{1+3\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^2}\frac{dz}{iz} = \oint_C \frac{1}{1+\frac{3}{4}\left(z+\frac{1}{z}\right)^2}\frac{1}{iz}\,dz$$

$$= \oint_C \frac{-i}{z+\frac{3}{4}z\left(z+\frac{1}{z}\right)^2}\,dz$$

$$= -i\oint_C \frac{1}{z+\frac{3}{4}z\left(z^2+2+\frac{1}{z}\right)}\,dz$$

$$= -i\oint_C \frac{1}{z+\frac{3}{4}\left(z^3+2z+\frac{1}{z}\right)}\,dz$$

$$= -i\oint_C \frac{1}{\frac{3}{4}z^3 + \frac{5}{2}z + \frac{3}{4z}}dz$$

$$= -i\oint_C \frac{4}{3z^3 + 10z + \frac{3}{z}}dz$$

$$= -4i\oint_C \frac{1}{3z^3 + 10z + \frac{3}{z}}dz$$

$$= -4i\oint_C \frac{z}{3z^4 + 10z^2 + 3}dz.$$

We use the Cauchy integral formula. Factorize the denominator:

$$= -4i\oint_C \frac{z}{3z^3 + 10z^2 + 3}dz$$

$$= -4i\oint_C \frac{z}{3(3z^2 + 3)(z^2 + 1/3)}dz$$

$$= -4i\oint_C \frac{z}{3\left(z + \sqrt{3i}\right)\left(z - \sqrt{3i}\right)\left(z + \sqrt{\frac{1}{3}}i\right)\left(z - \sqrt{\frac{1}{3}}i\right)}dz$$

$$= -\frac{4}{3}i\oint_C \frac{z}{\left(z + \sqrt{3i}\right)\left(z - \sqrt{3i}\right)\left(z + \sqrt{\frac{1}{3}}i\right)\left(z - \sqrt{\frac{1}{3}}i\right)}dz.$$

The singularities then to be considered are at $3^{-1/2}i$, $-3^{-1/2}i$. We can now reduce the integral:

$$= -\frac{4}{3}i\oint_{C_1}\frac{\dfrac{z}{(z+\sqrt{3i})(z-\sqrt{3i})\left(z+\sqrt{\dfrac{1}{3}}i\right)}}{\left(z-\sqrt{\dfrac{1}{3}}i\right)}dz$$

$$+-\frac{4}{3}i\oint_{C_2}\frac{\dfrac{z}{(z+\sqrt{3i})(z-\sqrt{3i})\left(z-\sqrt{\dfrac{1}{3}}i\right)}}{\left(z+\sqrt{\dfrac{1}{3}}i\right)}$$

where C_1 is a small circle about $3^{-1/2}i$, and C_2 is a small circle about $-3^{-1/2}i$. We can now apply the formula:

$$= -\frac{4}{3}i\left(2\pi i\left(\frac{z}{(z+\sqrt{3i})(z-\sqrt{3i})\left(z+\sqrt{\frac{1}{3}}i\right)}\right)\Bigg|_{z=\sqrt{\frac{1}{3}}i}\right.$$

$$\left.+2\pi i\left(\frac{z}{(z+\sqrt{3i})(z-\sqrt{3i})\left(z-\sqrt{\frac{1}{3}}i\right)}\right)\Bigg|_{z=-\sqrt{\frac{1}{3}}i}\right)$$

$$= -\frac{4}{3}i\left(2\pi i\left(\frac{\sqrt{\frac{1}{3}}i}{\left(\sqrt{\frac{1}{3}}i+\sqrt{3i}\right)\left(\sqrt{\frac{1}{3}}i-\sqrt{3i}\right)\left(\sqrt{\frac{1}{3}}i-\sqrt{\frac{1}{3}}i\right)}\right)\right.$$

$$\left.+2\pi i\left(\frac{-\sqrt{\frac{1}{3}}i}{\left(-\sqrt{\frac{1}{3}}i+\sqrt{3i}\right)\left(-\sqrt{\frac{1}{3}}i-\sqrt{3i}\right)\left(-\sqrt{\frac{1}{3}}i-\sqrt{\frac{1}{3}}i\right)}\right)\right)$$

$$= -\frac{4}{3}i\left(2\pi i\left(\frac{\sqrt{\frac{1}{3}}\,i}{\left(\frac{4}{\sqrt{3}}i\right)\left(-\frac{2}{\sqrt{3}}i\right)\left(\frac{2}{\sqrt{3}}i\right)}\right)+2\pi i\left(\frac{-\sqrt{\frac{1}{3}}\,i}{\left(\frac{2}{\sqrt{3}}i\right)\left(-\frac{4}{\sqrt{3}}i\right)\left(-\frac{2}{\sqrt{3}}i\right)}\right)\right)$$

$$= -\frac{4}{3}i\left(2\pi i\left(\frac{\sqrt{\frac{1}{3}}\,i}{i\left(\frac{4}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)}\right)+2\pi i\left(\frac{-\sqrt{\frac{1}{3}}\,i}{-i\left(\frac{2}{\sqrt{3}}\right)\left(\frac{4}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)}\right)\right)$$

$$= -\frac{4}{3}i\left(2\pi i\left(\frac{\sqrt{\frac{1}{3}}}{\left(\frac{4}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)}\right)+2\pi i\left(\frac{-\sqrt{\frac{1}{3}}}{\left(\frac{2}{\sqrt{3}}\right)\left(\frac{4}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)}\right)\right)$$

$$= -\frac{4}{3}i\left(2\pi i\left(\frac{\sqrt{\frac{1}{3}}}{\frac{16}{3\sqrt{3}}}\right)+2\pi i\left(\frac{\sqrt{\frac{1}{3}}}{\frac{16}{3\sqrt{3}}}\right)\right)$$

$$= -\frac{4}{3}i\left(2\pi i\left(\frac{3}{16}\right)+2\pi i\left(\frac{3}{16}\right)\right) = -\frac{4}{3}i\left(\pi i\left(\frac{3}{8}+\frac{3}{8}\right)\right) = \frac{4}{3}\left(\frac{3}{4}\right)\pi = \pi.$$

EXAMPLE (IV) - BRANCH CUTS

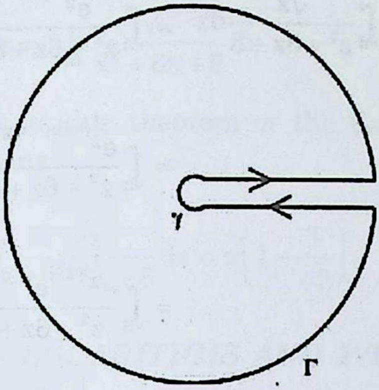
Consider

$$\int_0^\infty \frac{\sqrt{x}}{x^2+6x+8}\,dx.$$

We can begin by formulating the complex integral

$$\int_C \frac{\sqrt{z}}{z^2+6z+8}\,dz = I.$$

We can use the Cauchy integral formula or residue theorem again to obtain the relevant residues. However, the important thing to note is that $z^{1/2}=e^{1/2\,\mathrm{Log}(z)}$, so $z^{1/2}$ has a branch cut. This affects our choice of the contour C. Normally the logarithm branch cut is defined as the negative real axis, however, this makes the calculation of the integral slightly more complex, so we define it to be the positive real axis.



Then, we use the so-called keyhole contour, which consists of a small circle about the origin of radius e say, extending to a line segment parallel and close to the positive real axis but not touching it, to an almost full circle, returning to a line segment parallel, close, and below the positive real axis in the negative sense, returning to the small circle in the middle.

Let γ be the small circle of radius ε, Γ the larger, with radius r, then

$$\int_C \frac{\sqrt{z}}{z^2+6z+8}dz = \int_\varepsilon^R \frac{\sqrt{z}}{z^2+6z+8}dz$$

$$+ \int_\Gamma \frac{\sqrt{z}}{z^2+6z+8}dz$$

$$+ \int_R^\varepsilon \frac{\sqrt{z}}{z^2+6z+8}dz$$

$$+ \int_\gamma \frac{\sqrt{z}}{z^2+6z+8}dz$$

Since $z^{1/2}=e^{1/2\ \mathrm{Log}(z)}$, along the contour below the branch cut, we have gained 2π in argument along Γ, so

$$\int_R^{\epsilon}\frac{\sqrt{z}}{z^2+6z+8}dz = \int_R^{\epsilon}\frac{e^{\frac{1}{2}\log(z)}}{z^2+6z+8}dz$$

$$= \int_R^{\epsilon}\frac{e^{\frac{1}{2}(\log|z|+i\arg z)}}{z^2+6z+8}dz$$

$$= \int_R^{\epsilon}\frac{e^{\frac{1}{2}\log|z|}\,e^{1/2(2\pi i)}}{z^2+6z+8}dz$$

$$= \int_R^{\epsilon}\frac{e^{\frac{1}{2}\log|z|}\,e^{\pi i}}{z^2+6z+8}dz$$

$$= \int_R^{\epsilon}\frac{-\sqrt{x}}{x^2+6x+8}dx$$

$$= \int_{\epsilon}^{R}\frac{-\sqrt{x}}{x^2+6x+8}dx$$

simplifying,
$$= \int_{\epsilon}^{R}\frac{\sqrt{x}}{x^2+6x+8}dx,$$

and then
$$\int_C\frac{\sqrt{z}}{z^2+6z+8}dz = \int_{\epsilon}^{R}\frac{\sqrt{z}}{z^2+6z+8}dz$$

$$+ \int_{\Gamma}\frac{\sqrt{z}}{z^2+6z+8}dz$$

$$+ \int_{\epsilon}^{R}\frac{\sqrt{z}}{z^2+6z+8}dz$$

$$+ \int_{\gamma}\frac{\sqrt{z}}{z^2+6z+8}dz$$

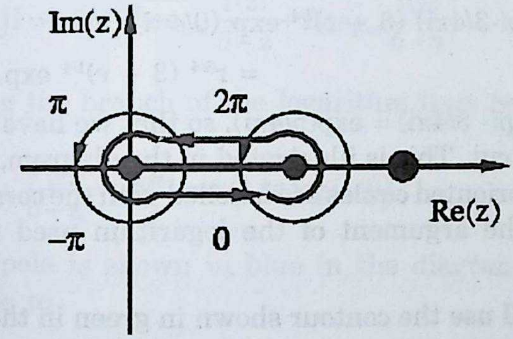
It can be shown that the integrals over Γ and γ both tend to zero as e tends to zero and R tends to infinity, by an estimation argument above. Thus, then,

$$\int_c \frac{\sqrt{z}}{z^2+6z+8}\,dz = 2\int_0^\infty \frac{\sqrt{x}}{x^2+6x+8}\,dx.$$

By using the residue theorem or the Cauchy integral formula one obtains

$$\pi i\left(\frac{i}{\sqrt{2}}-i\right) = \int_0^\infty \frac{\sqrt{x}}{x^2+6x+8}\,dx = \pi\left(1-\frac{1}{\sqrt{2}}\right).$$

EXAMPLE (V) - LOGARITHMS AND THE RESIDUE AT INFINITY



We seek to evaluate

$$I = \int_0^3 \frac{\left(x^3(3-x)\right)^{1/4}}{5-x}\,dx.$$

This requires a close study of

$f(z) = (z^3(3-z))^{1/4}$.

We will construct f(z) so that it has a branch cut on [0,3], shown in red in the diagram. To do this, we choose two branches of the logarithm, setting

$$(z^3)^{1/4} = z^{3/4} = \exp\ (3/4 \log\ (z))$$

where $-\pi \le \arg\ (\log(z) < \pi$

and

$$(3 - z)^{1/4} = \exp\ (1/4 \log\ (3 - z))$$

where $0 \le \arg\ (\log(z) < 2\pi$.

The cut of $z^{3/4}$ is therefore $(-\infty, 0]$ and the cut of $(3 - z)^{1/4}$ is $(-\infty, 3]$. It is easy to see that the cut of the product of the two, i.e. $f(z)$, is $[0,3]$, because $f(z)$ is actually continuous across $(-\infty, 0)$. This is because when $z = - r < 0$ and we approach the cut from above, $f(z)$ has the value

$$r^{3/4} \exp(-3/4\pi i)\ (3 + r)^{1/4} \exp\ (0/4\pi i)$$
$$= r^{3/4}\ (3 + r)^{1/4} \exp\ (5/4\pi i).$$

When we approach from below, $f(z)$ has the value

$$r^{3/4} \exp(-3/4\pi i)\ (3 + r)^{1/4} \exp\ (0/4\pi i)$$
$$= r^{3/4}\ (3 + r)^{1/4} \exp\ (-3/4\pi i).$$

But $\exp(- 3/4\pi i) = \exp(5/4\pi i)$, so that we have continuity across the cut. This is illustrated in the diagram, where the two black oriented circles are labelled with the corresponding value of the argument of the logarithm used in $z^{3/4}$ and $(3 - z)^{1/4}$.

We will use the contour shown in green in the diagram. To do this we must compute the value of $f(z)$ along the line segments just above and just below the cut. Let $z = r$ (in the limit, i.e. as the two green circles shrink to radius zero), where $0 \le r \le 3$. Along the upper segment, we find that $f(z)$ has the value

$$r^{3/4} \exp(3/4\ 0\ \pi i)\ (3 - r)^{1/4} \exp\ (2/4\pi i) = i r^{3/4}\ (3 - r)^{1/4}$$

and along the lower segment,

$$r^{3/4} \exp(3/4\ 0\ \pi i)\ (3 - r)^{1/4} \exp\ (0/4\pi i) = r^{3/4}\ (3 - r)^{1/4}$$

It follows that the integral of $f(z)/(5 - z)$ along the upper segment is $-i\ I$ in the limit, and along the lower segment, I.

If we can show that the integrals along the two green circles vanish in the limit, then we also have the value of I, by the Cauchy residue theorem. Let the radius of the green circles be ρ, where $\rho < 1/1000$ and $\rho \to 0$ and apply the ML-inequality. For the circle C_L on the left, we find

$$\left|\int_{CL}\frac{f(z)}{5-z}dz\right| \le 2\pi\rho\frac{\rho^{3/4}(3+1/1000)^{1/4}}{5-1/1000} \in O\left(\rho^{7/4}\right) \to 0.$$

Similarly, for the circle C_R on the right, we have

$$\left|\int_{CR}\frac{f(z)}{5-z}dz\right| \le 2\pi\rho\frac{(3+1/1000)^{3/4}\cdot\rho^{1/4}}{2-1/1000} \in O\left(\rho^{5/4}\right) \to 0.$$

Now using the Cauchy residue theorem, we have

$$(1-i)I = -2\pi i\left(\mathrm{Res}_{z=5}\frac{f(z)}{5-z} + \mathrm{Res}_{z=\infty}\frac{f(z)}{5-z}\right).$$

Using the branch of the logarithm from before, clearly

$$\mathrm{Res}_{z=5}\frac{f(z)}{5-z} = -5^{3/4}\exp\left(1/4\log(-2)\right).$$

The pole is shown in blue in the diagram. The value simplifies to

$-5^{3/4} \exp (1/4 (\log(2)+\pi i)) = -\exp (1/\pi i)\ 5^{3/4}\ 2^{1/4}$.

We use the following formula for the residue at infinity:

$$\mathrm{Res}_{z=\infty}h(z) = \mathrm{Res}_{z=0} -\frac{1}{z^2}h\left(\frac{1}{z}\right).$$

Substituting, we find

$1/(5-1/z) = -z(1 + 5z + 5^2z^2 + 5^3z^3 + \dots)$

and

$(1/z^3 (3 - 1/z))^{1/4} = 1/z\ (3z - 1)^{1/4}\ 1/z\ \exp (1/4\pi i)\ (1 - 3z)^{14}$, where we have used the fact that - 1 = exp(πi) for the second branch of the logarithm. Next we apply the binomial expansion, obtaining

$$1/z\ \exp(1/4\pi i)\left(\left(\frac{1/4}{1}\right)3z+\left(\frac{1/4}{2}\right)3^2z^2-\left(\frac{1/4}{3}\right)3^3z^3+\ldots\right).$$

The conclusion is that

$$\mathrm{Res}_{z=\infty}\frac{f(z)}{5-z}=\exp(1/4\pi i)(5-3/4)=\exp(1/4pi)17/4.$$

Finally, it follows that the value of I is

$$I=-2\pi i\frac{\exp(1/4\pi i)}{1-i}(17/4-5^{3/4}2^{1/4})$$

$$=-2\pi i 2^{-1/2}i(17/4-5^{3/4}2^{1/4})$$

which yields

$$I=\frac{\pi}{2\sqrt{2}}\left(17-5^{3/4}2^{9/4}\right)=\frac{\pi}{2\sqrt{2}}\left(17-40^{3/4}\right).$$

Partial Fractions in Integration

In integral calculus, the use of partial fractions is required to integrate the general rational function. Any rational function of a real variable can be written as the sum of a polynomial function and a finite number of partial fractions. Each partial fraction has as its denominator a polynomial function of degree 1 or 2, or some positive integer power of such a function. If the denominator is a 1st-degree polynomial or a power of such a polynomial, then the numerator is a constant. If the denominator is a 2nd-degree polynomial or a power of such a polynomial, then the numerator is a 1st-degree polynomial.

For an account of how to find this partial fraction expansion of a rational function, see partial fraction.

This article is about what to do after finding the partial fraction expansion, when one is trying to find the function's antiderivative.

A 1st-degree Polynomial in the Denominator

The substitution u = ax + b, du = a dx reduces the integral

$$\int \frac{1}{ax+b}dx$$

to

$$\int \frac{1}{u}\frac{du}{a} = \frac{1}{a}\int \frac{du}{u} = \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C.$$

A Repeated 1st-degree Polynomial in the Denominator

The same substitution reduces such integrals as

$$\int \frac{1}{(ax+b)^8}dx$$

to

$$\int \frac{1}{u^8}\frac{du}{a} = \frac{1}{2}\int u^{-8}du = \frac{1}{a}.\frac{u^{-7}}{(-7)} + C$$

$$= \frac{-1}{7au^7} + C = \frac{-1}{7z(ax+b)^7} + C.$$

An Irreducible 2nd-degree Polynomial in the Denominator

Next we consider such integrals as

$$\int \frac{x+6}{x^2-8x+25}dx.$$

The quickest way to see that the denominator $x^2 - 8x + 25$ is irreducible is to observe that its discriminant is negative. Alternatively, we can complete the square:

$$x^2 - 8x + 25 = (x^2 - 8x + 16) + 9 = (x - 4)^2 + 9$$

and observe that this sum of two squares can never be 0 while x is a real number.

In order to make use of the substitution

$u = x^2 - 8x + 25$

$du = (2x - 8)dx$

$du/2 = (x - 4)\ dx$

we would need to find x - 4 in the numerator. So we decompose the numerator x + 6 as (x - 4) + 10, and we write the integral as

$$\int \frac{x-4}{x^2-8x+25}dx + \int \frac{10}{x^2-8x+25}dx.$$

The substitution handles the first summand, thus:

$$\int \frac{x-4}{x^2-8x+25}dx = \int \frac{du/2}{u} = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln(x^2-8x+25) + C.$$

Note that the reason we can discard the absolute value sign is that, as we observed earlier, $(x - 4)^2 + 9$ can never be negative.

Next we must treat the integral

$$\int \frac{10}{x^2-8x+25}dx.$$

First, complete the square, then do a bit more algebra:

$$\int \frac{10}{x^2-8x+25}dx = \int \frac{10}{(x-4)^2+9}dx = \int \frac{10/9}{\left(\frac{x-4}{3}\right)^2+1}dx$$

Now the substitution

w = (x - 4)/3

dw = dx/3

gives us

$$\frac{10}{3}\int\frac{dw}{w^2+1}=\frac{10}{3}\,\arctan(w)+C=\frac{10}{3}\arctan\left(\frac{x-4}{3}\right)+C.$$

Putting it all together,

$$\int\frac{x+6}{x^2-8x+25}dx=\frac{1}{2}\ln\left(x^2-8x+25\right)+\frac{10}{3}\arctan\left(\frac{x-4}{3}\right)+C.$$

A Repeated Irreducible 2nd-degree Polynomial in the Denominator

Next, consider

$$\int\frac{x+6}{(x^2-8x+25)^8}dx.$$

Just as above, we can split x + 6 into (x - 4) + 10, and treat the part containing x - 4 via the substitution

u = x^2 - 8x + 25,

du = (2x - 8)dx

du/2 = (x - 4) dx

This leaves us with

$$\int\frac{x+6}{(x^2-8x+25)^8}dx.$$

As before, we first complete the square and then do a bit of algebraic massaging, to get

$$\int\frac{10}{(x^2-8x+25)^8}dx=\int\frac{10}{((x-4)^2+9)^8}dx=\int\frac{10/9^8}{\left(\left(\frac{x-4}{3}\right)^2+1\right)^8}dx.$$

Then we can use a trigonometric substitution:

$$\tan\theta = \frac{x-4}{3},$$

$$\left(\frac{x-4}{3}\right)^2 + 1 = \tan^2\theta + 1 = \sec^2\theta,$$

$$d\tan\theta = \sec^2\theta d\theta = \frac{dx}{3}.$$

Then the integral becomes

$$\int \frac{30/9^8}{\sec^{16}\theta}\sec^2\theta d\theta = \frac{30}{9^8}\int \cos^{14}\theta d\theta$$

By repeated applications of the half-angle formula

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

one can reduce this to an integral involving no higher powers of cos ? higher than the 1st power.

Then one faces the problem of expression sin and cos as functions of x. Recall that

$$\tan(\theta) = \frac{x-4}{3},$$

and that tangent = opposite/adjacent. If the "opposite" side has length x - 4 and the "adjacent" side has length 3, then the Pythagorean theorem tells us that the hypotenuse has length O((x - 4)^2 + 3^2) = O(x^2 -8x + 25).

Therefore we have

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x-4}{\sqrt{x^2 - 8x + 25}},$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{\sqrt{x^2 - 8x + 25}},$$

and

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \frac{6(x-4)}{x^2-8x+25}.$$

Auxiliary Fraction

In Swami Bharati Krishna Tirtha's Vedic mathematics, the auxiliary fraction method is used to convert a fraction to its equivalent decimal representation. The "auxiliary fraction" is not a true fraction, but is simply a mnemonic aid used in the calculation. The method is essentially the long division algorithm adapted for mental calculation. It is simplest when the fraction's denominator is one less than a multiple of 10, when it uses the identity

$$a = qb + r$$

$$\Rightarrow 10a = q\ (10b - 1) + (10r + q).$$

Variants of the method used when the denominator is not one less than a multiple of 10 become progressively more complex.

Dividing by Powers of Ten

To divide by a power of ten first move the decimal point in both the numerator and the denominator to the left the same number of places as the number of zeros at the end of the denominator. Then divide. For example, 1/800 = 0.01/8; 39/70 = 3.9/7; 3741/110000 = 0.3741/11; and 97654/90,000,000 = 0.0097654/9.

Forming the Auxiliary Fraction

The formation of the auxiliary fraction depends on the denominator. There are four cases:

- Type One A: denominators which end in a single nine
- Type One B: denominators which end in several nines

- Type Two: denominators which end in one
- Type Three: denominators which end in the digits, 2, 3, 4, 5, 6, 7, and 8.

If the denominator ends in zero(s), its first non-zero digit (from the right) identifies the family of the denominator.

Type One A

When the fraction's denominator ends in a single nine use the Ekadhika Purva. First replace the denominator by its Ekadhika which means adding one to the denominator and dividing the result by 10. Divide the numerator by 10 also. Hereafter, F is the actual fraction to be converted to a decimal value and A.F. is the auxiliary fraction used to remember the original dividend and the recurrent divisor.

EXAMPLES:

When F = 1/19, then the A.F. = 1/20 = 0.1/2;

When F = 4/29, then the A.F. = 4/30 = 0.4/3;

When F = 8/59, then the A.F. = 8/60 = 0.8/6;

Type One B

When the denominator ends in several nines, increase the denominator by one, then divide both numerator and denominator by a power of 10 equal to the number of terminal nines in the denominator.

EXAMPLES:

When F = 174/1299, then the A.F. = 174/1300 = 1.74/13;

When F = 1/6999, then the A.F. = 1/7000 = 0.001/7;

If the denominator ends in 1, 3, or 7, multiply both denominator and numerator by 9, 3 or 7 respectively to convert to an equivalent fraction in which the denominator ends in 9.

EXAMPLES:

When F = 36/121 = 324/1089, then the A.F. = 32.4/109;

When F = 53/93 = 159/279, then the A.F. = 15.9/28;

When F = 15/37 = 105/259, then the A.F. = 10.5/26.

Type Two

When the fraction has a denominator ending in one, form the auxiliary fraction by subtracting one from the denominator and from the numerator. Then divide both numerator and denominator by a power of 10 equal to the number of terminal zeros in the new denominator.

EXAMPLES:

When F = 3/61, then the A.F = 2/60 = 0.2/6;

When F = 28/71, then the A.F. = 27/70 = 2.7/7;

When F = 1/81, then the A.F. = 0/80 = 0.0/8;

When F = 14/131, then the A.F. = 13/130 = 1.3/13;

When F = 1/301, then the A.F. = 0/300 = 0.00/3;

When F = 6163/8001, then the

A.F. = 6162/8000 = 6.162/8;

If the denominator ends in 3 or 7, multiply both denominator and numerator by 7 or 3 respectively to convert to an equivalent fraction in which the denominator ends in 1.

EXAMPLES:

When F = 2/3 = 14/21, then the A.F. = 13/20 = 1.3/2;

When F = 10/27 = 30/81, then the A.F.= 29/80 = 2.9/8;

When F = 5/67 = 15/201, then the A.F.=14/200= 0.14/2;

When F = 4/17 = 12/51, then the A.F. = 11/50 = 1.1/5.

Type Three

For denominators that do not end in 1 or 9, use the nines family auxiliary fraction and count the number of units (above or below) that the ending is from the normal nine.

Using the Auxiliary Fraction — Type One

Given a fraction whose denominator ends in 9, first write the auxiliary fraction. Next divide in the auxiliary fraction to generate one (or more) quotient digit(s) at-a-time. Then write the quotient digit(s) and the remainder. The remainder at each step is prefixed to the just generated quotient digit for the next division. So if the quotient and remainder from one step are q and r, the dividend for the next step is 10r+q.

This algorithm meets the Vedic ideal of mental math with one line notation.

Word Examples — Type One A

EXAMPLE ONE

To convert the fraction F = 1/169 to a (repeating) decimal:

First, estimate the quotient as about six thousandths. Then, set up the auxiliary fraction, AF = 0.1/17. The first dividend is 0.1 and the working divisor is 17. Calculate one quotient digit at a time. Set down the remainder as a (subscripted) prefix to the quotient-digit just produced. Continue dividing to generate the quotient of the precision desired. Remember that the prefixed remainders are not parts of the quotient but only prefixes to the quotient-group in question and are dropped out of the answer — the lower row is a mere scaffolding and goes out.

As there are 168 remainders for the repeating decimal value of one one-hundred-sixty-ninth, there can be at most 168 digits before the decimal expansion repeats. If the remainder were to be zero or 169, then the fraction would terminate as an exact decimal value. There are in fact 78 digits before the decimal expansion of 1/169 repeats, and 78 is half of 156, which is the totient of 169.

17 into 0.1 goes 0 rem 1. 17 into 10 goes 0 rem 10. 17 into 100 goes 5 rem 15. 17 into 155 goes 9 rem 2. (The third dividend, 155, is formed by the remainder 15 prefixed to quotient from the previous step, 5.)

$F = 1/169 = {}_{.1}0.\ {}_{1}0\ {}_{10}0\ {}_{15}5\ {}_{2}9\ {}_{12}1\ {}_{2}7\ {}_{10}1\ {}_{16}5\ {}_{12}9\ {}_{10}7\ {}_{5}6\ {}_{5}3\ {}_{2}3\ {}_{6}1\ {}_{10}3\ {}_{1}6\ {}_{15}0\ {}_{7}9\ {}_{11}4\ {}_{12}6\ {}_{7}7 \ldots$

This all the notation that is needed. The calculating is mental using the multiples of 17: 17, 34, 51, 68, 85, 102, 119, 136, 153, 170.

$F = 1/169 \approx 0.00591715976331361\ldots$

EXAMPLE TWO

If the denominator is 73, this is a special case because (73)(137) = 10001. F = 1/73 = 137/10001 = (137)(9,999)/(99,999,999) = 0.0136,9863 repeating. Thus we may expect an eight-digit repeating decimal with complementary halves, i.e., the digits in the first half of the repeating decimal digit set are complements of nine for the digits of the second half.

EXAMPLE THREE

When F = 3/73 = 9/219, then the A.F. = 9/22. The working divisor is 22. The dividend at each step is created by prefixing the remainder to the quotient from the previous step.

$F = 0.{}_{9}0\ {}_{2}4\ {}_{2}1\ {}_{21}0\ {}_{12}9\ {}_{19}5\ {}_{19}8\ {}_{0}9\ {}_{9}0\ {}_{2}4\ {}_{2}1\ {}_{21}0\ldots$

$F = 3/73 \approx 0.041095890410\ldots$ The fraction repeats after eight decimal places. Furthermore, the eight digits have complementary halves (see Midy's theorem).

Worked Example — Type One B

When F = 53/799, then the A.F. = 0.53/8

The working divisor is 8. As the denominator has two nines, the division proceeds in steps of two dividend digits at a time, generating two quotient digits at each step. The remainder is prefixed to the pair of quotient digits from the previous step to create the dividend for the next step.

$F = 53/799 = 0.{}_{5}06\ {}_{2}63\ {}_{7}32\ {}_{4}91\ {}_{3}61\ {}_{1}45\ 18\ldots$

$F = 53/799 \approx 0.06633291614518\ldots$

Generating a Denominator with more Terminal Nines

An additional technique is available to find another A.F. with a smaller divisor. By a judicious choice of the multiplier one can produce an equivalent fraction with more nines in the denominator.

When F = 1/7 = 7/49, then the A.F. = 0.7/5; We use a multiplier of 7 to produce an equivalent fraction, 7/49. Look at the denominator of the equivalent fraction, 49. Consider the 4. To build the 4 to a nine we need to add 5 in the tens place. The 5th multiple of 7 ends in 5, so we may use 5 tens or 57 as the multiplier.

F = 1/7 = 57/399, and the A.F. = 0.57/4 (dividing in bundles of two digits).

We may proceed likewise, until we have F = 1/7 = 142,857/999,999 giving a bundle of six nines and an A.F. = 0.142857/1. This means we have the repeating decimal digit set on sight because when we divide this bundle of six digits (0.142857) by one and we have no remainder and this bundle repeats!

Using the Auxiliary Fraction — Type Two

After forming the auxiliary fraction, divide in the first step, but create the next dividend by prefixing the remainder not to the quotient digit, but to the quotient digit's complement from nine. So if the quotient and remainder from one step are q and r, the dividend for the next step is $10r + (9 - q)$.

When the fraction has a numerator of one, the auxiliary fraction will have a numerator of zero. Having a dividend of zero is not a problem because on the second step the complement of zero is nine, an adequate dividend.

Worked Example — Type Two

F = 13/31, then the A.F. = 12/30 = 1.2/3

The working divisor is 3. The lower row is the dividend.

F = 13/31 = 0. 4 1 9 3 5 4 8 3 8 7 0 9

Prefix R to complement of quotient:

05 28 10 16 14 25 11 26 21 02 29 20

F = 13/31 ≈ 0.419354838709...

Using the Auxiliary Fraction — Type Three

When the fraction has a denominator that does not end in 1 or 9 apply the Anurupya Sutra, whereby, after prefixing each remainder to the quotient-digit in question, add to (or subtract from) the dividend at each step, as many times the quotient-digit as the divisor (the denominator) is below (or above) the normal nine. The process can be wholly mental with practice.

Worked Examples - Type Three

EXAMPLE ONE

If the denominator is 68, as 68=(4)(17), we may expect two fixed digits and 16 repeating digits. When F = 15/68, the A.F. = 1.5/7 (As D ends in 8, one below the normal ending, 9, we add the quotient-digit to the dividend at each step). The working divisor is 7. Divide to achieve the desired precision.

F = 15/68, A.F. = 1.5/7

A prefixed rem: $F = 0.{}_{1}2\ {}_{0}2\ {}_{4}0\ {}_{5}5\ {}_{4}8\ {}_{0}8\ {}_{2}2\ {}_{3}3\ {}_{1}5\ {}_{6}2\ {}_{1}9\ {}_{0}4\ {}_{1}1\ {}_{5}1$

Plus the Q-digit: 2 2 0 5 8 8 2 3 5 2 9 4 1 1

7 into actual dividend: 14 04 40 60 56 16 24 36 20 64 28 08 12 52

F = 15/68 ≈ 0.22058823529411...

EXAMPLE TWO

When F = 163/275, A.F. = 16.3/28. (Since D ends in 5, four below the normal ending, 9, we add four times the Q-digit to the dividend at each step.) The working divisor is 28. Since D = 275 = $(5^2)(11)$, we may expect two fixed digits, then a two-digit repeater. Remember that every factor of 2, 5, or 10 in the denominator generates one fixed decimal digit. Multiples of 28: 28, 56, 84, 112, 140, 168, 196, 224, 252, 280.

Prefix the rem: F = 0. $_{23}5$ $_{3}9$ $_{19}2$ $_{4}7$ $_{19}2$ $_{4}7$

Plus four times the Q-digit: x 20 36 08 28 08 28

28 into the actual dividend:163 255 75 200 75 200 75

F = 163/275 ≈ 0.592727...

6

Methods of Computing Square Roots

Exponential Identity

Pocket calculators typically implement good routines to compute the exponential function and the natural logarithm, and then compute the square root of S using the identity

$$\sqrt{S} = e^{\frac{1}{2}\ln S}.$$

The same identity is used when computing square roots with logarithm tables or slide rules.

Rough Estimation

Many of the methods for calculating square roots require an initial seed value. If the initial value is too far from the actual square root, the calculation will be slowed down. It is therefore useful to have a rough estimate, which may be very inaccurate but easy to calculate.

One way of obtaining such an estimate for $\sqrt{S}$ is calculating 3^D, where D is the number of digits (to the left of the decimal point) in a positive real S. If S < 1, D is the negative of the number of zeros to the immediate right of the decimal point.

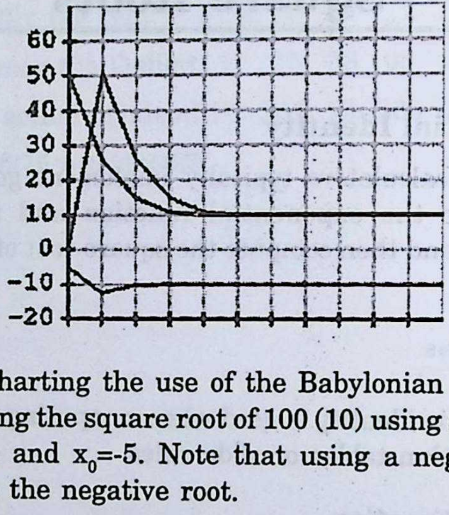
A better method of rough estimation is this:

If D is odd, D = 2n + 1, then use $\sqrt{S} \approx 2.10^n$

If D is even, D = 2n + 2, then use $\sqrt{S} \approx 6.10^n$.

When working in the binary numeral system (as computers do internally), an alternative method is to use $2^{[D/2]}$ (here D is the number of binary digits).

Babylonian Method



Graph charting the use of the Babylonian method for approximating the square root of 100 (10) using start values x_0=50, x_0=1, and x_0=-5. Note that using a negative start value yields the negative root.

A commonly used algorithm for approximating $\sqrt{S}$ (and perhaps the best) is known as the "Babylonian method" and can be derived from (but predates) Newton's method. This is a quadratically convergent algorithm, which means that the number of correct digits of the approximation roughly doubles with each iteration. It proceeds as follows:

1. Start with an arbitrary positive start value x_0 (the closer to the root, the better).
2. Let x_{n+1} be the average of xn and S/xn (using the arithmetic mean to approximate the geometric mean).

3. Repeat steps 2 and 3, until the desired accuracy is achieved.

It can also be represented as:

$$x_0 \approx \sqrt{S}.$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right),$$

$$\sqrt{S} = \lim_{n\to\infty} x_n.$$

This algorithm works equally well in the p-adic numbers, but cannot be used to identify real square roots with p-adic square roots; it is easy, for example, to construct a sequence of rational numbers by this method which converges to + 3 in the reals, but to - 3 in the 2-adics.

EXAMPLE

We'll calculate , where S = 125348, to 6 significant figures. Here D, the number of digits in S, is 6. Then:

$$x_0 = 3^D = 3^6 = 729.000$$

$$x_1 = \frac{1}{2}\left(x_0 + \frac{S}{x_0}\right) = \frac{1}{2}\left(729.000 + \frac{125348}{729.000}\right) = 450.472$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{S}{x_1}\right) = \frac{1}{2}\left(450.472 + \frac{125348}{450.472}\right) = 364.365$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{S}{x_2}\right) = \frac{1}{2}\left(364.365 + \frac{125348}{364.365}\right) = 354.191$$

$$x_4 = \frac{1}{2}\left(x_3 + \frac{S}{x_3}\right) = \frac{1}{2}\left(354.191 + \frac{125348}{354.191}\right) = 354.045$$

$$x_5 = \frac{1}{2}\left(x_4 + \frac{S}{x_4}\right) = \frac{1}{2}\left(354.045 + \frac{125348}{354.045}\right) = 354.045$$

Therefore, $\sqrt{125348} \approx 354.045.$

Convergence

We let the relative error in xn be defined by

$$\in_n = \frac{x_n}{\sqrt{S}} - 1$$

and thus

$$x_n = \sqrt{S}.(1 + \in_n).$$

Then one can show that

$$\in_{n+1} = \frac{\in_n^2}{2(1 + \in_n)}$$

and thus that

$$0 \leq \in_{n+1} \leq \min\left(\frac{\in_{n+1}^2}{2}, \frac{\in_{n+1}}{2}\right)$$

and consequently that convergence is assured provided that x0 and S are both positive.

Bakhshali Approximation

This is a method for finding an approximation to a square root which was described in an ancient manuscript known as the Bakhshali Manuscript. It is equivalent to two iterations of the Babylonian method beginning with N. The original presentation goes as follows: To calculate $\sqrt{S}$, let N^2 be the nearest perfect square to S. Then, calculate:

$$d = S - N^2$$

$$P = \frac{d}{2N}$$

$$A = N + P$$

$$\sqrt{S} \approx A - \frac{P^2}{2A}$$

This can be also written as:

$$\sqrt{S} \approx \frac{N^4 + 6N^2S + S^2}{4N^3 + 4NS}$$

$$\sqrt{N^2 + d} \approx N + \frac{d}{2N} - \frac{d^2}{8N^3 + 4Nd}$$

EXAMPLE

We'll find

$N = 3$

$d = 9.2345 - 3^2 = 0.2345$

$$P = \frac{0.2345}{2.3} = 0.0391$$

$A = 3 + 0.0391 = 3.0391$

$$\sqrt{9.2345} \approx 3.0391 - \frac{0.0391^2}{2 \times 3.0391} \approx 3.0388.$$

Digit by Digit Calculation

This is a method to find each digit of the square root in a sequence. It is much slower than the Babylonian method (if you have a calculator which can divide in one operation), but it has several advantages:

- It can be easier for manual calculations.
- Every digit of the root found is known to be correct, i.e. it will not have to be changed later.
- If the square root has an expansion which terminates, the algorithm will terminate after the last digit is found. Thus, it can be used to check whether a given integer is a square number.

Napier's bones include an aid for the execution of this algorithm. The Shifting nth-root algorithm is a generalization of this method.

The algorithm works for any base, and naturally, the way it proceeds depends on the base chosen. We will describe the method for the decimal system. It goes as follows:

Write the decimal expansion of the square. The numbers are laid out in a fashion similar to the long division algorithm, and the root will appear above its square. Divide the digits of the square into pairs, starting from the decimal point and going both left and right.

The decimal point of the root should be placed above the decimal point of the square. One digit of the root will appear above each pair of digits of the square.

Begin at the left-most (non-zero) pair of digits of the square. Initialize the remainder to zero. Initialize p to zero.

For each iteration (pair of digits) do:

1. Bring down the most significant (leftmost) pair of digits of the square not yet used (if all the digits have been used, use the digits 00) and append them to the remainder from the previous step, i.e. multiply the remainder by one hundred and add the two digits. This will be the current value c.
2. Find p, y and x, as follows:
    - Let p be the part of the root found so far, ignoring any decimal point. (For the first step, p = 0).
    - Let y equal $(20 . p + x) . x$.
    - Determine the greatest digit x such that $(20 . p + x) . x$ (that is, y) does not exceed c.
        - * Note: 20p + x is simply twice p, with the digit x appended to the right).
        - * Note: You can find x by guessing what c/(20·p) is and doing a trial calculation of y, then adjusting x upward or downward as necessary.

- Place the digit x as the next digit of the root, i.e. above the two digits of the square which you just brought down. Thus the next p will be the old p times 10 plus x.

3. Subtract y from c to form a new remainder.
4. If the remainder is zero and there are no more digits to bring down, then the algorithm has terminated. Otherwise go back to step 1 for another iteration.

EXAMPLES

Find the square root of 152.2756.

$$\begin{array}{l} \ \ 1\ \ \ 2.\ \ 3\ \ \ 4 \\ \sqrt{01\ \ 52.\ 27\ \ 56} \end{array}$$

x	01	1*1 <= 1 < 2*2	x = 1
	01	y = x*x = 1*1 = 1	
2x	00 52	22*2 <= 52 < 23*3	x = 2
	00 44	y = 2x*x = 22*2 = 44	
24x	08 27	243*3 <= 827 < 244*4	x = 3
	07 29	y = 24x*x = 243*3 = 729	
246x	98 56	2464*4 <= 9856 < 2465*5	x = 4
	98 56	y = 246x*x = 2464*4 = 9856	
	00 00	Algorithm terminates: Answer is 12.34	

144582

Find the square root of 2.

$$\begin{array}{l} \ \ \ \ 1.\ \ 4\ \ 1\ \ 4\ \ 2 \\ \sqrt{02.\ 00\ 00\ 00\ 00} \end{array}$$

x	02	1*1 <= 2 < 2*2	x = 1
	01	y = x*x = 1*1 = 1	
2x	01 00	24*4 <= 100 < 25*5	x = 4
	00 96	y = 2x*x = 24*4 = 96	

28x	04 00	281*1 <= 400 < 282*2	x = 1
	02 81	y = 28x*x = 281*1 = 281	
282x	01 19 00	2824*4 <= 11900 < 2825*5	x = 4
	01 12 96	y = 282x*x = 2824*4 = 11296	
2828x	06 04 00	28282*2 <= 60400 < 28283*3	x = 2
		The desired precision is achieved:	
		The square root of 2 is about 1.4142	

Duplex Method for Extracting a Square Root

The duplex method is a variant of the digit by digit method for calculating the square root of a whole or decimal number one digit at a time. The duplex is the square of the central digit plus double the cross-product of digits equidistant from the centre. The duplex is computed from the quotient digits (square root digits) computed thus far, but after the initial digits. The duplex is subtracted from the dividend digit prior to the second subtraction for the product of the quotient digit times the divisor digit. For perfect squares the duplex and the dividend will get smaller and reach zero after a few steps. For non-perfect squares the decimal value of the square root can be calculated to any precision desired. However, as the decimal places proliferate, the duplex adjustment gets larger and longer to calculate. The duplex method follows the Vedic ideal for an algorithm, one-line, mental calculation. It is flexible in choosing the first digit group and the divisor. Small divisors are to be avoided by starting with a larger initial group.

In short, to calculate the duplex of a number, double the product of each pair of equidistant digits plus the square of the centre digit (of the digits to the right of the colon).

Number ⇒ Calculation = Duplex

574 ⇒ 2(5·4) + 72 = 89

406,739 ⇒ 2(4·9)+ 2(0·3)+ 2(6·7) = 72+0+84 = 156

123,456 ⇒ 2(1·6)+ 2(2·5)+ 2(3·4) = 12 +20 +24 = 56

88,900,777 ⇒ 2(56)+2(56)+2(63)+0+0 = 320+30 = 350

48329,03711 ⇒ 2(4·1)+2(8·1)+2(3·7)+2(2·3)+2(9·0)

= 8+16+42+12+0 = 78

In a square root calculation the quotient digit set increases incrementally for each step.

Number ⇒ Calculation = Duplex:

1 ⇒ 12 = 1

14 ⇒ 2(1·4) = 8

142 ⇒ 2(1·2) + 42 = 4 + 16 = 20

14,21 ⇒ 2(1·1) + 2(4·2) = 2 + 16 = 18

14213 ⇒ 6+8+4 = 18

142,135 ⇒ 10+24+4 = 38

1421356 ⇒ 12+40+12+1 = 65

1421,3562 ⇒ 4+48+20+6 = 78

142,135,623 ⇒ 6+16+24+10+9 = 65

142,1356,237 ⇒ 14+24+8+12+30 = 88

142,13562,373 ⇒ 6+56+12+4+36+25 = 139

EXAMPLE 1

Consider $53^2 = 2809$.

- Set down the number in groups of two digits.
- We define a divisor, a dividend and a quotient to find the root.
- Consider the first group. Here is it 28.
  - Find the nearest perfect square below that group. Here, for the 28, the closest perfect square is 25.
  - The root of that perfect square is the first digit of our root.

  - Since 28 > 25 and 25 = 52, we take 5 as the first digit in the root.
  - For the divisor we take double this first digit (2 · 5), which is here 10.
- We set up a division framework with a colon.
  - 28: 0 9 is the dividend and 5: is the quotient.
  - We put a colon to the right of 28 and 5 and keep the colons lined up vertically. The duplex is calculated only on quotient digits to the right of the colon.
- We calculate the remainder. 28: minus 25: is 3:.
  - We append the remainder to the next digit to get the new dividend.
  - Here, we append 0 to the digit 3, which makes the new dividend 30, as the divisor 10 goes into 30 just 3 times. (No reserve needed here for subsequent deductions.)
- We repeat the operation.
  - The zero remainder appended to 9. Nine is the dividend.
  - Now we have a digit to the right of the colon so we deduct the duplex. 32 = 9.
  - We get a zero remainder.
  - The next root digit is zero. The next duplex is 2(3·0) = 0.
  - The dividend is zero. We have an exact square root, 53.0.

Find the square root of 2809.

Set down the number in groups of two digits.

The number of groups gives the number of digits in the root.

From the first group, 28, we obtain the divisor, 10.

Put a colon after the first group to separate it.

Since 28>25=52. And 2x5=10.

Gross dividend: 28: 0 9. Using mental math:

Divisor: 10) 3 0 Square: 10) 28: $_3$0 9

Duplex, Deduction: 25: xx 09 Square root: 5: 3. 0

Dividend: 30 00

Remainder: 3: 00 00

Square Root, Quotient: 5: 3. 0

EXAMPLE 2

Find the square root of 2,080,180,881. Solution by the duplex method: this ten-digit square has five digit-pairs, so it will have a five-digit square root. The first digit-pair is 20. Put the colon to the right. The nearest square below 20 is 16, whose root is 4, the first root digit. So, we use 2·4=8 for the divisor. Now we proceed with the duplex division, one digit column at a time. Prefix the remainder to the next dividend digit.

divisor; gross dividend:	8) 20:	8	0	1	8	0	8	8	1
read the dividend diagonally up:		4	8	7	11	10	10	0	8
minus the duplex:	16:	xx	25	60	36	90	108	00	81
actual dividend:	:	48	55	11	82	10	00	08	00
minus the product:	:	40	48	00	72	00	00	0	00
remainder:	4:	8	7	11	10	10	0	8	00
quotient:	4:	5,	6	0	9.	0	0	0	0

Duplex calculations:

Quotient-digits ⇒ Duplex deduction.

5 ⇒ 52= 25

5 and 6 ⇒ 2(5·6) = 60

5,6,0 ⇒ 2(5·0)+62 = 36

$5,6,0,9 \Rightarrow 2(5\cdot 9)+2(6\cdot 0) = 90$

$5,6,0,9,0 \Rightarrow 2(5\cdot 0)+2(6\cdot 9)+ 0 = 108$

$5,6,0,9,0,0 \Rightarrow 2(5\cdot 0)+2(6\cdot 0)+2(0\cdot 9) = 0$

$5,6,0,9,0,0,0 \Rightarrow 2(5\cdot 0)+2(6\cdot 0)+2(0\cdot 0)+92 = 81$

Hence the square root of 2,080,180,881 is exactly 45,609.

EXAMPLE 3

Find the square root of two to ten places. Let us take 20,000 as the beginning group, using three digit-pairs at the start. The perfect square just below 20,000 is 141, since $141^2 = 19881 < 20,000$. So, the first root digits are 141 and the divisor doubled, 2 x 141 = 282. With a larger divisor the duplex will be relatively small. Hence, we can pick the multiple of the divisor without confusion.

Dividend:	2.0000 :	0	0	0	0	0	0	0	0
Diagonal;Divisor:	282) :	119	62	40	102	162	182	75	112
Minus duplex::	:	xxxx	16	16	12	28	53	74	59
Actual dividend:	20000 :	1190	604	384	1008	1592	1767	676	1061
Minus product:	19881 :	1128	564	282	846	1410	1692	564	846
Remainder:	119 :	62	40	102	162	182	75	112	215
Root quotient:	1.41 :	4	2	1	3	5	6	2	3

Ten multiples of 282: 282; 564; 846; 1128; 1410; 1692; 1974; 2256; 2538; 2820.

Iterative Methods for Reciprocal Square Roots

The following are iterative methods for finding the reciprocal square root of S which is $\frac{1}{\sqrt{S}}$. Once it has been found, we can find $\sqrt{S}$ by simple multiplication: $\sqrt{S} = \frac{1}{\sqrt{S}}.S.$ These iterations involve only multiplication, and not division. They are therefore faster than the Babylonian method. However, they are not stable. If the initial value is not close to the reciprocal square root, the iterations will diverge away from it rather than converge to it. It can therefore be advantageous to perform an iteration of the Babylonian method on a rough estimate before starting to apply these methods.

- One method is found by applying Newton's method to the equation $\frac{1}{x^2} - S = 0$. It converges quadratically:

  $$x_{n+1} = \frac{x_n}{2}.\left(3 - S.x_n^2\right).$$

- Another method converges cubically, but involves more operations per iteration:

  $$y_n = S.x_n^2,$$

  $$x_{n+1} = \frac{x_n}{8}.\left(15 - y_n.\left(10 - 3.y_n\right)\right).$$

Taylor Series

If N is an approximation to $\sqrt{S}$, a better approximation can be found by using the Taylor series of the square root function:

$$\sqrt{N^2 + d} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!\, d^n}{(1-2n)\, n!^2\, 4^n N^{2n-1}}$$

$$= N + \frac{d}{2N} - \frac{d^2}{8N^3} + \frac{d^3}{16N^5} - \frac{5d^4}{128N^7} + \dots$$

As an iterative method, the order of convergence is equal to the number of terms used. With 2 terms, it is identical to the Babylonian method; With 3 terms, each iteration takes almost as many operations as the Bakhshali approximation, but converges more slowly. Therefore, this is not a particularly efficient way of calculation.

Other Methods

Finding $\sqrt{S}$ is the same as solving the equation $x^2 - S = 0$. Therefore, any general numerical root-finding algorithm can be used. Newton's method, for example, reduces in this case to the Babylonian method. Other methods are less efficient than the ones presented above.

Continued Fraction Expansion

Quadratic irrationals (numbers of the form $\frac{a+\sqrt{b}}{c}$, where a, b and c are integers), and in particular, square roots of integers, have periodic continued fractions. Sometimes we may be interested not in finding the numerical value of a square root, but rather in its continued fraction expansion. The following iterative algorithm can be used for this purpose (S is any natural number which is not a perfect square):

$$m_0 = 0$$

$$d_0 = 1$$

$$a_0 = \left[\sqrt{S}\right]$$

$$m_{n+1} = d_n a_n - m_n$$

$$d_{n+1} = \frac{S - m_{n+1}^2}{d_n}$$

$$a_{n+1} = \left[\frac{\sqrt{S} + m_{n+1}}{d_{n+1}}\right] = \left[\frac{a_0 + m_{n+1}}{d_{n+1}}\right]$$

Notice that mn, dn, and an are always integers. The algorithm terminates when this triplet is the same as one encountered before. The expansion will repeat from then on. The sequence $[a_0; a_1, a_2, a_3, \ldots]$ is the continued fraction expansion:

$$\sqrt{S} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

Example, Square Root of 114 as a Continued Fraction

We begin with $m_0=0$; $d_0=1$; and $a_0=10$.

$$\sqrt{114} = \frac{\sqrt{114}+0}{1} = 10 + \frac{\sqrt{114}-10}{1}$$

$$= 10 + \frac{\left(\sqrt{114}-10\right)\left(\sqrt{114}+10\right)}{\sqrt{114}+10}$$

$$= 10 + \frac{114-100}{\sqrt{114}+10} = 10 + \cfrac{1}{\cfrac{\sqrt{114}+10}{14}}.$$

Next, $m_1=10$; $d_1=14$; and $a_1=1$.

$$\frac{\sqrt{114}+10}{14} = 1 + \frac{\sqrt{144}-4}{14}$$

$$= 1 + \frac{114-16}{14\left(\sqrt{144}-4\right)} = 1 + \cfrac{1}{\cfrac{\sqrt{114}+4}{7}}.$$

Next, $m_2=4$; $d_2=7$; and $a_2=2$.

$$\frac{\sqrt{114}+4}{7} = 2 + \frac{\sqrt{114}-10}{7}$$

$$= 2 + \frac{14}{7\left(\sqrt{114}+10\right)} = 2 + \cfrac{1}{\cfrac{\sqrt{114}+10}{2}}.$$

$$\frac{\sqrt{114}+10}{2} = 10 + \frac{\sqrt{114}-10}{2}$$

$$= 10 + \frac{14}{2\left(\sqrt{114}+10\right)} = 10 + \cfrac{1}{\cfrac{\sqrt{114}+10}{7}}.$$

$$\frac{\sqrt{114}+10}{7}=2+\frac{\sqrt{114}-4}{7}$$

$$=2+\frac{98}{7\left(\sqrt{114}+10\right)}=2+\frac{1}{\frac{\sqrt{114}+4}{14}}.$$

$$\frac{\sqrt{114}+4}{7}=1+\frac{\sqrt{114}-10}{14}$$

$$=1+\frac{14}{14\left(\sqrt{114}+10\right)}=1+\frac{1}{\frac{\sqrt{114}+10}{1}}.$$

$$\frac{\sqrt{114}+10}{1}=20+\frac{\sqrt{114}-10}{1}$$

$$=20+\frac{14}{\sqrt{114}+10}=20+\frac{1}{\frac{\sqrt{114}+10}{14}}.$$

Now, loop back to the second equation above.

Consequently, the continued fraction for the square root of 114 is:

$\sqrt{114}$ = [10; 1, 2, 10, 2, 1, 20, 1, 2, 10, 2, 1, 20, 1, 2, 10, 2, 1, 20, ...].

Pell's Equation

Pell's equation and its variants yield a method for efficiently finding continued fraction convergents of square roots of integers. However, it can be complicated to execute, and usually not every convergent is generated. The ideas behind the method are as follows:

- If (p, q) is a solution (where p and q are integers) to the equation $p^2 = S.q^2 \pm 1$, then $\frac{p}{q}$ is a continued fraction convergent of $\sqrt{S}$, and as such, is an excellent rational approximation to it.

- If (p_a, q_a) and (p_b, q_b) are solutions, then so is:
  $p = p_a p_b + S.q_a q_b$
  $q = p_a q_b + p_b q_a$
- More generally, if (p_1, q_1) is a solution, then it is possible to generate a sequence of solutions (p_n, q_n) satisfying:
  $p_{m+n} = p_m p_n + S.q_m q_n$
  $q_{m+n} = q_m q_n + p_n q_m$

The method is as follows:

- Find natural numbers (0 excluded) p1 and q1 such that $p_1^2 = S.q_1^2 \pm 1$. This is the hard part; It can be done either by guessing, or by using fairly sophisticated techniques.
- To generate a long list of convergents, iterate:
  $p_{n+1} = p_1 p_n + S.q_1 q_n$
  $q_{n+1} = p_1 q_n + p_n q_1$
- To find the larger convergents quickly, iterate:
  $p_{2n} = p_n^2 + S.q_n^2$
  $q_{2n} = 2p_n q_n$
- In either case, is a rational approximation satisfying

$$\left|\frac{p_n}{q_n} - \sqrt{S}\right| < \frac{1}{q_n^2.\sqrt{S}}.$$

Approximations that Depend on IEEE Representation

On computers, a very rapid Newton's method based approximation to the square root can be obtained for floating point numbers when computers use an IEEE (or sufficiently similar) representation.

```
float fastsqrt(float val) {
    int tmp = *(int *)&val;
    tmp-= 1<<23;/*Remove last bit to not let it go to mantissa */
    /* tmp is now an approximation to logbase2(val) */
    tmp = tmp >> 1;/* divide by 2 */
    tmp += 1<<29;/*add 64 to exponent: (e+127)/2=(e/2)+63,*/
                 /*that represents (e/2)-64 but we want e/2*/
    return *(float *)&tmp;
}
```

In the above, the operations to remove last exponent bit and add the IEEE bias can be combined into a single operation. An additional adjustment can be made in the same operation to reduce the maximum relative error. So, the three operations, not including the cast, can be rewritten as:

```
     tmp = (1<<29) + (tmp >> 1) - (1<<22) + m;
```

Where m is some magic number that corresponds to a calculation involving an initial guess used in the Newton's approximation.

Inverse Square Root

A variant of the above routine is included below, which can be used to compute the reciprocal of the square root, i.e. instead, was written by Greg Walsh, and implemented into the game Quake 3 by Gary Tarolli. As an approximation, it produced a relative error of less than 4%, and in computer graphics it is a very efficient way to normalize a vector.

```
float invSqrt(float x)
{
    float xhalf = 0.5f*x;
    int i = *(int *)&x;
    i = 0x5f3759df - (i >> 1);
    x = *(float *)&i;
    x = x * (1.5f - xhalf * x * x);
    return x;
}
```

The Vedic Mathematics Sutras

This list of sutras is taken from the book Vedic Mathematics, which includes a full list of the sixteen Sutras in Sanskrit, but in some cases a translation of the Sanskrit is not given in the text and comes from elsewhere.

This formula 'On the Flag' is not in the list given in Vedic Mathematics, but is referred to in the text.

The Main Sutras

By one more than the one before.
All from 9 and the last from 10.
Vertically and Crosswise
Transpose and Apply
If the Samuccaya is the Same it is Zero
If One is in Ratio the Other is Zero
By Addition and by Subtraction
By the Completion or Non-Completion
Differential Calculus
By the Deficiency
Specific and General
The Remainders by the Last Digit
The Ultimate and Twice the Penultimate
By One Less than the One Before
The Product of the Sum
All the Multipliers

The Sub Sutras

Proportionately
The Remainder Remains Constant
The First by the First and the Last by the Last
For 7 the Multiplicand is 143

By Osculation
Lessen by the Deficiency
Whatever the Deficiency lessen by that amount andset up the Square of the Deficiency
Last Totalling 10
Only the Last Terms
The Sum of the Products
By Alternative Elimination and Retention
By Mere Observation
The Product of the Sum is the Sum of the Products
On the Flag

Try a Sutra

Mark Gaskell introduces an alternative system of calculation based on Vedic philosophy

At the Maharishi School in Lancashire we have developed a course on Vedic mathematics for key stage 3 that covers the national curriculum.

The results have been impressive: maths lessons are much livelier and more fun, the children enjoy their work more and expectations of what is possible are very much higher. Academic performance has also greatly improved: the first class to complete the course managed to pass their GCSE a year early and all obtained an A grade.

Vedic maths comes from the Vedic tradition of India. The Vedas are the most ancient record of human experience and knowledge, passed down orally for generations and written down about 5,000 years ago.

Medicine, architecture, astronomy and many other branches of knowledge, including maths, are dealt with in the texts. Perhaps it is not surprising that the country credited with introducing our current number system and the invention of perhaps the most important mathematical symbol, 0, may have more to offer in the field of maths.

The remarkable system of Vedic maths was rediscovered from ancient Sanskrit texts early last century. The system is based on 16 sutras or aphorisms, such as: "by one more than the one before" and "all from nine and the last from 10". These describe natural processes in the mind and ways of solving a whole range of mathematical problems.

For example, if we wished to subtract 564 from 1,000 we simply apply the sutra "all from nine and the last from 10". Each figure in 564 is subtracted from nine and the last figure is subtracted from 10, yielding 436.

1,000 -	564	= 436		
1,000 -	5	6	4	
	subtract from 9	subtract from 9	subtract from 10	
	↓	↓	↓	
	4	3	6	

This can easily be extended to solve problems such as 3,000 minus 467. We simply reduce the first figure in 3,000 by one and then apply the sutra, to get the answer 2,533. We have had a lot of fun with this type of sum, particularly when dealing with money examples, such as £10 take away £2. 36. Many of the children have described how they have challenged their parents to races at home using many of the Vedic techniques — and won. This particular method can also be expanded into a general method, dealing with any subtraction sum.

The sutra "vertically and crosswise" has many uses. One very useful application is helping children who are having trouble with their tables above 5x5. For example 7x8. 7 is 3 below the base of 10, and 8 is 2 below the base of 10.

7 × 8 = 56

7 3 (3 is the difference from base)

8 2 (2 is the difference from base

(A) 7 3 starting at the left subtract crosswise either 8-3 or 7-2 to get 5, the first figure of the answer

8 2

5

(B) 7 3 Multiply vertically

× to get 6

8 2

5 6

The whole approach of Vedic maths is suitable for slow learners, as it is so simple and easy to use.

The sutra "vertically and crosswise" is often used in long multiplication. Suppose we wish to multiply

32 by 44. We multiply vertically 2×4=8.

Then we multiply crosswise and add the two results: 3×4+4×2=20, so put down 0 and carry 2.

Finally we multiply vertically 3×4=12 and add the carried 2 =14. Result: 1,408.

We can extend this method to deal with long multiplication of numbers of any size. The great advantage of this system is that the answer can be obtained in one line and mentally. By the end of Year 8, I would expect all students to be able to do a "3 by 2" long multiplication in their heads. This gives enormous confidence to the pupils who lose their fear of numbers and go on to tackle harder maths in a more open manner.

All the techniques produce one-line answers and most can be dealt with mentally, so calculators are not used until Year 10. The methods are either "special", in that they only apply under certain conditions, or general. This encourages flexibility and innovation on the part of the students.

Multiplication can also be carried out starting from the left, which can be better because we write and pronounce numbers from left to right. Here is an example of doing this in a special method for long multiplication of numbers near a base (10, 100, 1,000 etc.), for example, 96 by 92. 96 is 4 below the base and 92 is 8 below.

We can cross-subtract either way: 96-8=88 or 92-4=88. This is the first part of the answer and multiplying the "differences" vertically 4×8=32 gives the second part of the answer.

96 × 96 = 8,832

(A) 96 4 (4 is the difference from base)
92 8 (8 is the difference from base

(B) 96 4 subtract crosswise from the
left 96-8 = 88
92 8 o 92-4 = 88

88

(C) 96 4 Multiply vertically
× 4 × 8 = 32
92 8

---

88 32

This works equally well for numbers above the base: 105x111=11,655. Here we add the differences. For

205x211=43,255, we double the first part of the answer, because 200 is 2x100.

We regularly practise the methods by having a mental test at the beginning of each lesson. With the introduction of a non-calculator paper at GCSE, Vedic maths offers methods that are simpler, more efficient and more readily acquired than conventional methods.

There is a unity and coherence in the system which is not found in conventional maths. It brings out the beauty and patterns in numbers and the world around us. The techniques are so simple they can be used when conventional methods would be cumbersome.

When the children learn about Pythagoras's theorem in Year 9 we do not use a calculator; squaring numbers and finding square roots (to several significant figures) is all performed with relative ease and reinforces the methods that they would have recently learned.

TUTORIAL 1

Use the formula ALL FROM 9 AND THE LAST FROM 10 to perform instant subtractions.

For example 1000 - 357 = 643

We simply take each figure in 357 from 9 and the last figure from 10.

So the answer is 1000 - 357 = 643

And thats all there is to it!

This always works for subtractions from numbers consisting of a 1 followed by noughts: 100; 1000; 10,000 etc.

Similarly 10,000 - 1049 = 8951

For 1000 - 83, in which we have more zeros than figures in the numbers being subtracted, we simply suppose 83 is 083.

So 1000 - 83 becomes 1000 - 083 = 917

EXERCISE 1 TUTORIAL 1

Try some yourself:

1) 1000 - 777 =
2) 1000 - 283 =
3) 1000 - 505 =
4) 10,000 - 2345 =
5) 10,000 - 9876 =
6) 10,000 - 1011 =
7) 100 - 57 =
8) 1000 - 57 =
9) 10,000 - 321 =
10) 10,000 - 38 =

TUTORIAL 2

Using VERTICALLY AND CROSSWISE you do not need the multiplication tables beyond 5 × 5.

Suppose you need 8 × 7

8 is 2 below 10 and 7 is 3 below 10.

Think of it like this:

8 2
7 3
5 6 answer

The answer is 56.

The diagram below shows how you get it.

8 2
× |
7 3
5 6 answer

You subtract crosswise 8-3 or 7 - 2 to get 5, the first figure of the answer.

And you multiply vertically: 2 × 3 to get 6, the last figure of the answer.

That's all you do:

See how far the numbers are below 10, subtract one number's deficiency from the other number, and multiply the deficiencies together.

7 × 6 = 42

$$\begin{array}{cc} 7 & 3 \\ \times & | \\ 6 & 4 \\ \hline 3 & 12 = 42 \\ \hline \end{array}$$

Here there is a carry: the 1 in the 12 goes over to make 3 into 4.

EXERCISE 1 TUTORIAL 2

Multiply These:

1) 8 × 8 =

2) 9 × 7 =

3) 8 × 9 =

4) 7 × 7 =

5) 9 × 9 =

6) 6 × 6 =

Here's how to use VERTICALLY AND CROSSWISE for multiplying numbers close to 100.

Suppose you want to multiply 88 by 98.

Not easy, you might think. But with VERTICALLY AND CROSSWISE you can give the answer immediately, using the same method as above.

Both 88 and 98 are close to 100.

88 is 12 below 100 and 98 is 2 below 100.

You can imagine the sum set out like this:

$$\begin{array}{rr} 88 & -12 \\ \times & | \\ 98 & -\ 2 \\ \hline 86 & 24 \\ \hline \end{array}$$

As before the 86 comes from subtracting crosswise: 88 - 2 = 86 (or 98 - 12 = 86: you can subtract either way, you will always get the same answer).

And the 24 in the answer is just 12 × 2: you multiply vertically.

So 88 × 98 = 8624

EXERCISE 2 TUTORIAL 2

This is so easy it is just mental arithmetic.

Try some:

1) 87 × 98 =
2) 88 × 97 =
3) 77 × 98 =
4) 93 × 96 =
5) 94 × 92 =
6) 64 × 99 =
7) 98 × 97 =

ANSWERS TO EXERCISE 2 TUTORIAL 2

1) 8526
2) 8536
3) 7546
4) 8928
5) 8648

6) 6336
7) 9506 (we put 06 because, like all the others, we need two figures in each part)

Multiplying numbers just over 100.

103 × 104 = 10712

The answer is in two parts: 107 and 12,

107 is just 103 + 4 (or 104 + 3), and 12 is just 3 × 4.

Similarly 107 × 106 = 11342

107 + 6 = 113 and 7 × 6 = 42

EXERCISE 3 TUTORIAL 2

Again, just for mental arithmetic

Try a few:

1) 102 × 107 =
2) 106 × 103 =
3) 104 × 104 =
4) 109 × 108 =
5) 101 × 123 =
6) 103 × 102 =

ANSWERS TO EXERCISE 3 TUTORIAL 2

1) 10914
2) 10918
3) 10816
4) 11772
5) 12423
6) 10506 (we put 06, not 6)

TUTORIAL 3

The easy way to add and subtract fractions.

1) $\frac{4}{5}+\frac{1}{6}=$

2) $\frac{1}{3}+\frac{1}{4}=$

3) $\frac{2}{7}+\frac{2}{3}=$

4) $\frac{4}{5}-\frac{1}{6}=$

Use VERTICALLY AND CROSSWISE to write the answer straight down!

$$\frac{2}{3}+\frac{1}{5}=\frac{10+3}{15}=\frac{13}{15}$$

Multiply crosswise and add to get the top of the answer:

2 × 5 = 10 and 1 × 3 = 3. Then 10 + 3 = 13.

The bottom of the fraction is just 3 × 5 = 15.

You multiply the bottom number together.

So:

$$\frac{5}{7}+\frac{3}{4}=\frac{20+21}{28}=\frac{41}{28}$$

Subtracting is just as easy: multiply crosswise as before, but the subtract:

$$\frac{6}{7}-\frac{2}{3}=\frac{18-14}{21}=\frac{4}{21}$$

TUTORIAL 4

A quick way to square numbers that end in 5 using the formula BY ONE MORE THAN THE ONE BEFORE.

$75^2 = 5625$

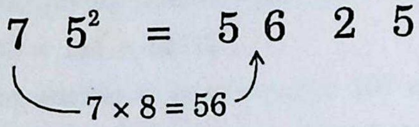
75^2 means 75 × 75.

The answer is in two parts: 56 and 25.

The last part is always 25.

The first part is the first number, 7, multiplied by the number "one more", which is 8:

so 7 × 8 = 56



Similarly $85^2 = 7225$ because 8 × 9 = 72.

EXERCISE 1 TUTORIAL 4

Try these:

1) 452 =

2) 652 =

3) 952 =

4) 352 =

5) 152 =

Method for multiplying numbers where the first figures are the same and the last figures add up to 10.

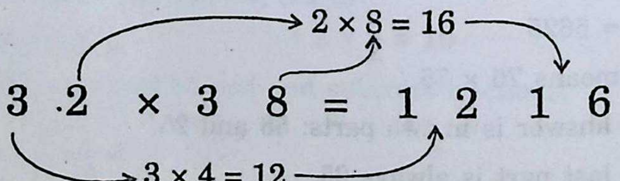
32 × 38 = 1216

Both numbers here start with 3 and the last figures (2 and 8) add up to 10.

So we just multiply 3 by 4 (the next number up) to get 12 for the first part of the answer.

And we multiply the last figures: 2 × 8 = 16 to get the last part of the answer.

Diagrammatically:



And 81 × 89 = 7209

We put 09 since we need two figures as in all the other examples.

ANSWERS TO EXERCISE 1 TUTORIAL 4

1) 2025
2) 4225
3) 9025
4) 1225
5) 225

EXERCISE 2 TUTORIAL 4

Practise some:

1) 43 × 47 =
2) 24 × 26 =
3) 62 × 68 =
4) 17 × 13 =
5) 59 × 51 =
6) 77 × 73 =

ANSWERS TO EXERCISE 2 TUTORIAL 4

1) 2021
2) 624
3) 4216
4) 221
5) 3009
6) 5621

TUTORIAL 5

An elegant way of multiplying numbers using a simple pattern

21 × 23 = 483

This is normally called long multiplication butactually the answer can be written straight downusing the VERTICALLY AND CROSSWISE formula.

We first put, or imagine, 23 below 21:

$$\begin{array}{cccl} 2 & & 1 & \\ | & \times & | & \\ 2 & & 3 & \times \\ \hline 4 & 8 & 3 & \\ \hline \end{array}$$

There are 3 steps:

a) Multiply vertically on the left: 2 × 2 = 4.

This gives the first figure of the answer.

b) Multiply crosswise and add: 2 × 3 + 1 × 2 = 8

This gives the middle figure.

c) Multiply vertically on the right: 1 × 3 = 3

This gives the last figure of the answer.

And thats all there is to it.

Similarly 61 × 31 = 1891

$$\begin{array}{cccl} 6 & & 1 & \\ | & \times & | & \\ 3 & & 1 & \times \\ \hline 18 & 9 & 1 & \\ \hline \end{array}$$

6 × 3 = 18; 6 × 1 + 1 × 3 = 9; 1 × 1 = 1

EXERCISE 1 TUTORIAL 5

Try these, just write down the answer:

1) 14 × 21
2) 22 × 31
3) 21 × 31
4) 21 × 22
5) 32 × 21

ANSWERS TO EXERCISE 1 TUTORIAL 5

1) 294
2) 682
3) 651
4) 462
5) 672

EXERCISE 2A TUTORIAL 5

Multiply any 2-figure numbers together by mere mental arithmetic!

If you want 21 stamps at 26 pence each you can easily find the total price in your head.

There were no carries in the method given above./p>

However, there only involve one small extra step.

21 × 26 = 546

2 1
| × |
2 6 ×
4 4 6 = 546

The method is the same as above except that we get a 2-figure number, 14, in the middle step, so the 1 is carried over to the left (4 becomes 5).

So 21 stamps cost £5.46.

Practise a few:

1) 21 × 47
2) 23 × 43
3) 32 × 53
4) 42 × 32
5) 71 × 72

ANSWERS TO EXERCISE 2A TUTORIAL 5

1) 987
2) 989
3) 1696
4) 1344
5) 5112

EXERCISE 2B TUTORIAL 5

33 × 44 = 1452

There may be more than one carry in a sum:

$$\begin{array}{ccccc} 3 & & 3 & & \\ | & \times & | & & \\ 4 & & 4 & \times & \\ \hline 12 & {}_{2}4 & {}_{1}2 & & = 1452 \\ \hline \end{array}$$

Vertically on the left we get 12.

Crosswise gives us 24, so we carry 2 to the left and mentally get 144.

Then vertically on the right we get 12 and the 1 here is carried over to the 144 to make 1452.

6) 32 × 56
7) 32 × 54
8) 31 × 72
9) 44 × 53
10) 54 × 64

ANSWERS TO EXERCISE 2B TUTORIAL 5

6) 1792
7) 1728
8) 2232
9) 2332
10) 3456

TUTORIAL 6

Multiplying a number by 11.

To multiply any 2-figure number by 11 we just put the total of the two figures between the 2 figures.

26 × 11 = 286

Notice that the outer figures in 286 are the 26 being multiplied.

And the middle figure is just 2 and 6 added up.

So 72 × 11 = 792

EXERCISE 1 TUTORIAL 6

Multiply by 11:

1) 43 =
2) 81 =
3) 15 =
4) 44 =
5) 11 =

ANSWERS TO EXERCISE 1 TUTORIAL 6

1) 473
2) 891
3) 165
4) 484
5) 121

EXERCISE 2 TUTORIAL 6

Multiply by 11:

1) 11 × 88 =
2) 11 × 84 =
3) 11 × 48 =
4) 11 × 73 =
5) 11 × 56 =

ANSWERS TO EXERCISE 2 TUTORIAL 6

1) 968
2) 924
3) 528
4) 803
5) 616

EXERCISE 3 TUTORIAL 6

Multiply by 11:

1) 151 =
2) 527 =
3) 333 =
4) 714 =
5) 909 =

ANSWERS TO EXERCISE 3 TUTORIAL 6

1) 1661
2) 5797
3) 3663
4) 7854
5) 9999

TUTORIAL 7

Method for dividing by 9.

23/9 = 2 remainder 5

The first figure of 23 is 2, and this is the answer.

The remainder is just 2 and 3 added up!

43/9 = 4 remainder 7

The first figure 4 is the answer and 4 + 3 = 7 is the remainder - could it be easier?

EXERCISE 1A TUTORIAL 7

Divide by 9:

1) 61/9 =
2) 33/9 =
3) 44/9 =
4) 53/9 =
5) 80/9 =

ANSWERS TO EXERCISE 1A TUTORIAL 7

134/9 = 14 remainder 8

The answer consists of 1,4 and 8.

1 is just the first figure of 134.

4 is the total of the first two figures 1+ 3 = 4, and 8 is the total of all three figures 1+ 3 + 4 = 8.

ANSWERS TO EXERCISE 1A TUTORIAL 7

1) 6 r 7
2) 3 r 6
3) 4 r 8
4) 5 r 8
5) 8 r 8

EXERCISE 1B TUTORIAL 7

Divide by 9:

6) 232 =
7) 151 =
8) 303 =
9) 212 =
10) 2121 =

ANSWERS TO EXERCISE 1B TUTORIAL 7

1) 25 r 7

2) 16 r 7
3) 33 r 6
4) 23 r 5
5) 235 r 6 (we have 2, 2 + 1, 2 + 1 + 2, 2 + 1 + 2 + 1)

EXERCISE 2 TUTORIAL 7

Divide these by 9:

1) 771/9 =
2) 942/9 =
3) 565/9 =
4) 555 /9 =
5) 2382/9 =
6) 7070/9 =

Is it Divisible by Four?

This little math trick will show you whether a number is divisible by four or not.

So, this is how it works.

Let's look at 1234

Does 4 divide evenly into 1234?

For 4 to divide into any number we have to make sure that the last number is even

If it is an odd number, there is no way it will go in evenly.

So, for example, 4 will not go evenly into 1233 or 1235

Now we know that for 4 to divide evenly into any number the number has to end with an even number.

4 into 1234, the solution:

Take the last number and add it to 2 times the second last number

If 4 goes evenly into this number then you know that 4 will go evenly into the whole number.

So,

4 + (2 × 3) = 10

4 goes into 10 two times with a remainder of 2 so it does not go in evenly.

Therefore 4 into 1234 does not go in completely.

Let's try 4 into 3436546

So, from our example, take the last number, 6 and add it to two times the penultimate number, 4

6 + (2 × 4) = 14

4 goes into 14 three times with two remainder.

So it doesn't go in evenly.

Let's try one more.

4 into 212334436

6 + (2 × 3) = 12

4 goes into 12 three times with 0 remainder.

Therefore 4 goes into 234436 evenly.

So what use is this trick to you?

Well if you have learnt the tutorial at Memorymentor.com about telling the day in any year, then you can use it in working out whether the year you are calculating is a leap year or not.

Multiplying by 12 - shortcut

So how does the 12's shortcut work?

Let's take a look.

12 × 7

The first thing is to always multiply the 1 of the twelve by the number we are multiplying by, in this case 7.

So 1 × 7 = 7.

Multiply this 7 by 10 giving 70. (Why? We are working with BASES here.

Bases are the fundamentals to easy calculations for all multiplication tables.

Now multiply the 7 by the 2 of twelve giving 14. Add this to 70 giving 84.

Therefore 7 × 12 = 84

Let's try another:

17 × 12

Remember, multiply the 17 by the 1 in 12 and multiply by 10

(Just add a zero to the end)

1 × 17 = 17, multiplied by 10 giving 170.

Multiply 17 by 2 giving 34.

Add 34 to 170 giving 204.

So 17 × 12 = 204

lets go one more

24 × 12

Multiply 24 × 1 = 24. Multiply by 10 giving 240.

Multiply 24 by 2 = 48. Add to 240 giving us 288

24 × 12 = 288 (these are Seriously Simple Sums to do aren't they?!)

Converting Kilos to Pounds

In this section you will learn how to convert Kilos to Pounds, and Vice Versa.

Let's start off with looking at converting Kilos to pounds.

86 kilos into pounds:

Step one, multiply the kilos by TWO.

To do this, just double the kilos.

86 × 2 = 172

Step two, divide the answer by ten.

To do this, just put a decimal point one place in from the right.

172/10 = 17.2

Step three, add step two's answer to step one's answer.

172 + 17.2 = 189.2

86 Kilos = 189.2 pounds

Let's try:

50 Kilos to pounds:

Step one, multiply the kilos by TWO.

To do this, just double the kilos.

50 × 2 = 100

Step two, divide the answer by ten.

To do this, just put a decimal point one place in from the right.

100/10 = 10

Step three, add step two's answer to step one's answer.

100 + 10 = 110

50 Kilos = 110 pounds

Adding Time

Here is a nice simple way to add hours and minutes together:

Let's add 1 hr and 35 minutes and 3 hr 55 minutes together.

What you do is this: make the 1 hr 35 minutes into one number, which will give us 135 and do the same for the other number, 3 hours 55 minutes, giving us 355

Now you want to add these two numbers together:

135
355

490

So we now have a sub total of 490.

What you need to do to this and all sub totals is add the time constant of 40.

No matter what the hours and minutes are, just add the 40 time constant to the sub total.

490 + 40 = 530

So we can now see our answer is 5 hrs and 30 minutes!

Temperature Conversions

This is a shortcut to convert Fahrenheit to Celsius and vice versa.

The answer you will get will not be an exact one, but it will give you an idea of the temperature you are looking at.

Fahrenheit to Celsius:

Take 30 away from the Fahrenheit, and then divide the answer by two.

This is your answer in Celsius.

EXAMPLE:

74 Fahrenheit - 30 = 44. Then divide by two, 22 Celsius.

So 74 Fahrenheit = 22 Celsius.

Celsius to Fahrenheit just do the reverse:

Double it, and then add 30.

30 Celsius double it, is 60, then add 30 is 90

30 Celsius = 90 Fahrenheit

Remember, the answer is not exact but it gives you a rough idea.

Decimals Equivalents of Fractions

With a little practice, it's not hard to recall the decimal equivalents of fractions up to 10/11!

First, there are 3 you should know already:

1/2 = .5

1/3 = .333...

1/4 = .25

Starting with the thirds, of which you already know one:

1/3 = .333...

2/3 = .666...

You also know 2 of the 4ths, as well, so there's only one new one to learn:

1/4 = .25

2/4 = 1/2 = .5

3/4 = .75

Fifths are very easy. Take the numerator (the number on top), double it, and stick a decimal in front of it.

1/5 = .2

2/5 = .4

3/5 = .6

4/5 = .8

There are only two new decimal equivalents to learn with the 6ths:

1/6 = .1666...

2/6 = 1/3 = .333...

3/6 = 1/2 = .5

4/6 = 2/3 = .666...

5/6 = .8333...

What about 7ths? We'll come back to them at the end. They're very unique.

8ths aren't that hard to learn, as they're just smaller steps than 4ths.

If you have trouble with any of the 8ths, find the nearest 4th, and add .125 if needed:

1/8 = .125

2/8 = 1/4 = .25

3/8 = .375

4/8 = 1/2 = .5

5/8 = .625

6/8 = 3/4 = .75

7/8 = .875

9ths are almost too easy:

1/9 = .111...

2/9 = .222...

3/9 = .333...

4/9 = .444...

5/9 = .555...

6/9 = .666...

7/9 = .777...

8/9 = .888...

10ths are very easy, as well. Just put a decimal in front of the numerator:

1/10 = .1

2/10 = .2

3/10 = .3

4/10 = .4

5/10 = .5

6/10 = .6

7/10 = .7

8/10 = .8

9/10 = .9

Remember how easy 9ths were? 11th are easy in a similar way, assuming you know your multiples of 9:

1/11 = .090909...

2/11 = .181818...

3/11 = .272727...

4/11 = .363636...

5/11 = .454545...

6/11 = .545454...

7/11 = .636363...

8/11 = .727272...

9/11 = .818181...

10/11 = .909090...

As long as you can remember the pattern for each fraction, it is quite simple to work out the decimal place as far as you want or need to go!

Oh, I almost forgot! We haven't done 7ths yet, have we?

One-seventh is an interesting number:

1/7 = .142857142857142857...

For now, just think of one-seventh as: .142857

See if you notice any pattern in the 7ths:

1/7 = .142857...

2/7 = .285714...

3/7 = .428571...

4/7 = .571428...

5/7 = .714285...

6/7 = .857142...

Notice that the 6 digits in the 7ths ALWAYS stay in the same order and the starting digit is the only thing that changes!

If you know your multiples of 14 up to 6, it isn't difficult to,br> work out where to begin the decimal number. Look at this:

For 1/7, think "1 * 14", giving us .14 as the starting point.

For 2/7, think "2 * 14", giving us .28 as the starting point.

For 3/7, think "3 * 14", giving us .42 as the starting point.

For 4/14, 5/14 and 6/14, you'll have to adjust upward by 1:

For 4/7, think "(4 * 14) + 1", giving us .57 as the starting point.

For 5/7, think "(5 * 14) + 1", giving us .71 as the starting point.

For 6/7, think "(6 * 14) + 1", giving us .85 as the starting point.

Practice these, and you'll have the decimal equivalents of everything from 1/2 to 10/11 at your finger tips!

If you want to demonstrate this skill to other people, and you know your multiplication tables up to the hundreds for each number 1-9, then give them a calculator and ask for a 2-digit number (3-digit number, if you're up to it!) to be divided by a 1-digit number.

If they give you 96 divided by 7, for example, you can think, "Hmm... the closest multiple of 7 is 91, which is 13 * 7, with 5 left over. So the answer is 13 and 5/7, or: 13.7142857!"

Converting Kilometres to Miles

This is a useful method for when travelling between imperial and metric countries and need to know what kilometres to miles are.

The formula to convert kilometres to miles is number of (kilometres/8) × 5

So lets try 80 kilometres into miles

80/8 = 10

multiplied by 5 is 50 miles!

Another example

40 kilometres

40/8 = 5

5 × 5= 25 miles

Taxila University

Takshashila, (later corrupted as Taxila), one of the topmost centres of education at that time in India became Chanakya's breeding ground of acquiring knowledge in the practical and theoretical aspect. The teachers were highly knowledgeable who used to teach sons of kings. It is said that a certain teacher had 101 students and all of them were princes! The niversity at Taxila was well versed in teaching the subjects using the best of practical knowledge acquired by the teachers. The age of entering the university was sixteen. The branches of studies most sought after in around India ranged from law, medicine, warfare and other indigenous forms of learning. The four Vedas, archery, hunting, elephant-lore and 18 arts were taught at the university of Taxila. So prominent was the place where Chanakya received his education that it goes to show the making of the genius. The very requirements of admission filtered out the outlawed and people with lesser credentials.

At a time when the Dark Ages were looming large, the existence of a university of Taxila's grandeur really makes India stand apart way ahead of the European countries who struggled with ignorance and total information blackout. For the Indian subcontinent Taxila stood as a light house of higher knowledge and pride of India. In the present day world, Taxila is situated in Pakistan at a place called Rawalpindi. The university accommodated more than 10,000 students at a time. The university offered courses spanning a period of more than eight years. The students were admitted after graduating from their own countries. Aspiring students opted for elective subjects going for in depth studies in specialized branches of learning. After graduating from the university, the students are recognized as the best scholars in the subcontinent. It became a cultural heritage as time passed. Taxila was the junction where people of different origins mingled with each other and exchanged knowledge of their countries.

The university was famous as "Taxila" university, named after the city where it was situated. The king and rich people of the region used to donate lavishly for the development of the university. In the religious scriptures also, Taxila is mentioned as the place where the king of snakes, Vasuki selected Taxila for the dissemination of knowledge on earth.

Here it would be essential to mention briefly the range of subjects taught in the university of Taxila. (1) Science, (2) Philosophy, (3) Ayurveda, (4) Grammar of various languages, (5) Mathematics, (6) Economics, (7) Astrology, (8) Geography, (9) Astronomy, (10) Surgical science, (11) Agricultural sciences, (12) Archery and Ancient and Modern Sciences.

The university also used to conduct researches on various subjects.

Mathematics

Zero -The Most Powerful Tool

India invented the Zero, without which there would be no binary system. No computers! Counting would be clumsy and cumbersome! The earliest recorded date, an inscription of Zero on Sankheda Copper Plate was found in Gujarat, India (585-586 CE). In Brahma-Phuta-Siddhanta of Brahmagupta (7th century CE), the Zero is lucidly explained and was rendered into Arabic books around 770 CE. From these it was carried to Europe in the 8th century. However, the concept of Zero is referred to as Shunya in the early Sanskrit texts of the 4th century BCE and clearly explained in Pingala's Sutra of the 2nd century.

Geometry

Invention of Geometry

The word Geometry seems to have emerged from the Indian word 'Gyaamiti' which means measuring the Earth (land). And the word Trigonometry is similar to 'Trikonamiti' meaning measuring triangular forms. Euclid is credited with the invention of Geometry in 300 BCE while the concept of Geometry in India emerged in 1000 BCE, from the practice of making fire altars in square and rectangular shapes. The treatise of Surya Siddhanta (4th century CE) describes amazing details of Trigonometry, which were introduced to Europe 1200 years later in the 16th century by Briggs.

The Value of PI in India

The ratio of the circumference and the diameter of a circle are known as Pi, which gives its value as 3,1428571. The old Sanskrit text Baudhayana Shulba Sutra of the 6th century BCE mentions this ratio as approximately equal to 3. Aryabhatta in 499, CE worked the value of Pi to the fourth decimal place as 3.1416. Centuries later, in 825 CE Arab mathematician Mohammed Ibna Musa says that "This value has been given by the Hindus (Indians)".

Pythagorean Theorem or Baudhayana Theorem?

The so-called Pythagoras Theorem - the square of the hypotenuse of a right-angled triangle equals the sum of the square of the two sides - was worked out earlier in India by Baudhayana in Baudhayana Sulba Sutra. He describes: "The area produced by the diagonal of a rectangle is equal to the sum of the area produced by it on two sides."

[Note: Greek writers attributed the theorem of Euclid to Pythagoras]

The Decimal

100BCE the Decimal system flourished in India

> *"It was India that gave us the ingenious method of expressing all numbers by means of ten symbols (Decimal System)....a profound and important idea which escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity." -La Place*

Raising 10 to the Power of 53

The highest prefix used for raising 10 to a power in today's maths is 'D' for 10 to a power of 30 (from Greek Deca). While, as early as 100 BCE Indian Mathematicians had exact names for figures upto 10 to the power of 53.

ekam =1

dashakam =10

shatam =100 (10 to the power of 10)

sahasram =1000 (10 power of 3)

dashasahasram =10000 (10 power of 4)

lakshaha =100000 (10 power of 5)
dashalakshaha =1000000 (10 power of 6)
kotihi =10000000 (10 power of 7)
ayutam =1000000000 (10 power of 9)
niyutam = (10 power of 11)
kankaram = (10 power of 13)
vivaram = (10 power of 15)
paraardhaha = (10 power of 17)
nivahaaha = (10 power of 19)
utsangaha = (10 power of 21)
bahulam = (10 power of 23)
naagbaalaha = (10 power of 25)
titilambam = (10 power of 27)
vyavasthaana
pragnaptihi = (10 power of 29)
hetuheelam = (10 power of 31)
karahuhu = (10 power of 33)
hetvindreeyam = (10 power of 35)
samaapta lambhaha = (10 power of 37)
gananaagatihi) = (10 power of 39)
niravadyam = (10 power of 41)
mudraabaalam = (10 power of 43)
sarvabaalam = (10 power of 45)
vishamagnagatihi = (10 power of 47)
sarvagnaha = (10 power of 49)
vibhutangamaa = (10 power of 51)
tallaakshanam = (10 power of 53)
(In Anuyogdwaar Sutra written in 100 BCE one numeral is raised as high as 10 to the power of 140).

Astronomy

Indian astronomers have been mapping the skies for 3500 years.

1000 Years Before Copernicus

Copernicus published his theory of the revolution of the Earth in 1543. A thousand years before him, Aryabhatta in 5th century (400-500 CE) stated that the Earth revolves around the sun, "just as a person travelling in a boat feels that the trees on the bank are moving, people on earth feel that the sun is moving". In his treatise Aryabhatteeam, he clearly states that our earth is round, it rotates on its axis, orbits the sun and is suspended in space and explains that lunar and solar eclipses occur by the interplay of the sun, the moon and the earth.

The Law of Gravity - 1200 Years Before Newton

The Law of Gravity was known to the ancient Indian astronomer Bhaskaracharya. In his Surya Siddhanta, he notes:

"Objects fall on earth due to a force of attraction by the earth, therefore, the earth, the planets, constellations, the moon and the sun are held in orbit due to this attraction".

It was not until the late 17th century in 1687, 1200 years later, that Sir Isaac Newton rediscovered the Law of Gravity.

Measurement of Time

In Surya Siddhanta, Bhaskaracharya calculates the time taken for the earth to orbit the sun to 9 decimal places.

Bhaskaracharya = 365.258756484 days.

Modern accepted measurement = 365.2596 days.

Between Bhaskaracharya's ancient measurement 1500 years ago and the modern measurement the difference is only 0.00085 days, only 0.0002%.

34000TH of a Second to 4.32 Billion Years

India has given the idea of the smallest and the largest measure of time.

Krati Krati = 34,000th of a second

1 Truti = 300th of a second

2 Truti = 1 Luv

2 Luv = 1 Kshana

30 Kshana = 1 Vipal

60 Vipal = 1 Pal

60 Pal = 1 Ghadi (24 minutes)

2.5 Gadhi = 1 Hora (1 hour)

24 Hora = 1 Divas (1 day)

7 Divas = 1 saptaah (1 week)

4 Saptaah = 1 Maas (1 month)

2 Maas = 1 Rutu (1 season)

6 Rutu = 1 Varsh (1 year)

100 Varsh = 1 Shataabda (1 century)

10 Shataabda = 1 sahasraabda

432 Sahasraabda = 1 Yug (Kaliyug)

2 Yug = 1 Dwaaparyug

3 Yug = 1 Tretaayug

4 Yug = 1 Krutayug

10 Yug = 1 Mahaayug (4,320,000 years)

1000 Mahaayug = 1 Kalpa

1 Kalpa = 4.32 billion years

Merely 16 Sutras and 13 Sub-Sutra form the basis of Vedic Mathematics. These can be used to solve all types of Mathematical Problems!!!

These sutras long lost in the cryptic Vedas where the mathematical solutions were hidden in Sanskrit Verses for centuries. Sri Bharati Krsna Tirthaji's "Vedic Mathematics" recently discovered these techniques again and given it to all of us to use.

The 16 sutras, or rules, and 14 sub-sutras can be used to solve complex mathematical problems mentally, leaving your fellow students and colleagues in awe.

The 16 Sutras of Vedic Mathematics:

Sutra 1 : By one more than the one before

Sutra 2: All from 9 and the last from 10

Sutra 3 : Vertically and crosswise

Sutra 4 : Transpose and apply

Sutra 5 : If the Samuccaya is the same it is zero

Sutra 6 : If one is in ratio the other is zero

Sutra 7: By addition and by subtraction

Sutra 8: By the completion or non-completion

Sutra 9: Differential calculus

Sutra 10: By the deficiency

Sutra 11: Specific and general

Sutra 12: The remainders by the last digit

Sutra 13: The ultimate and twice the penultimate

Sutra 14: By one less than the one before

Sutra 15: The product of the sum

Sutra 16: All the multipliers

The 13 Sub-sutras:

SS 1: Proportionately

SS 2: The remainder remains constant

SS 3: The first by the first and the last by the last

SS 4: For 7 the multiplicand is 143

SS 5: By osculation

SS 6: Lessen by the deficiency

SS 7: Whatever the deficiency lessen by that amount and set up the square of the deficiency

SS 8: Last totalling 10

SS 9: Only the last terms

SS 10: The sum of the products

SS 11: By alternative elimination and retention

SS 12: By mere observation

SS 13: The product of the sum is the sum of the products

We will work through these Sutras in the next few pages and also will give you lot of practice to get used to the techniques.

Sutra 1: "By one more than the one before" provides a simple way of multiplication of numbers with the same first digit and the sum of their last unit digits is 10.

An interesting application of this formula is in computing squares of numbers ending in five. Consider:

$35 \times 35 = ((3 \times 3) + 3)*10 + 25 = 1225$ or

$= (3 \times 4) *10 +(5 \times 5) = 1225$

$125 \times 125 = (12 \times 13)*10 + 25 = 15625$

EXPLANATION

The latter portion is multiplied by itself (5 by 5) and the previous portion is multiplication of first digit and the digit higher to the first digit resulting in the answer 1225

This is a simple application of $(a+b)^2 = a^2 + 2ab + b^2$ ....

Here $2b = 10$

It can also be applied in multiplications when the last digit is not 5 but the sum of the last digits is the base (10) and the previous parts are the same. Consider:

$37 \times 33 = (3 \times 4)*100 + (7 \times 3) = 1221$

$29 \times 21 = (2 \times 3)*100 + (9 \times 1) = 609$

This uses $(a + b)(a - b) = a^2 - b^2$ twice combined with the previous result to produce:

$$\begin{aligned}(10c + 5 + d)(10c + 5 - d) &= (10c + 5)^2 - d^2\\ &= 100c(c + 1) + 25 - d^2\\ &= 100c(c + 1) + (5 + d)(5 - d).\end{aligned}$$

The Sutra is very useful in its application to convert fractions into their equivalent decimal form. Consider fraction 1/19. Using this formula, this can be converted into a decimal form in a single step. This can be done by applying the formula for either a multiplication or division operation, thus yielding two methods.

METHOD 1

Using multiplication. The sutra "one more than the one before" provides a simple way of calculating values like 1/x9 (e.g 1/19, 1/29, etc.). Let's take one 1/x9 and calculate e.g. 1/19. In this case, x=1. To convert 1/19 to decimals, since 19 is not divisible by 2 or 5, the fractional result is a purely circulating (recurring) decimal, i.e the digits would repeat themselves after some time. (If the denominator contains only factors 2 and 5 it is a purely non-circulating decimal i.e it is a perfect decimal, else it is a mixture of the two.).

So to get the decimals of 1/19, we start with the numerator digit i.e 1 in this case and Multiply this by "one more" (1+x), in this case: 1+1 = 2.

21 Multiplying 2 by 2, followed by multiplying 4 by 2

421 → 8421 Now, multiplying 8 by 2, sixteen

68421.1 ← carry

If the result of the multiplication is greater than 10, keep (value - 10) and keep the "1" as "carry over" which you'll add to the next digit.

multiplying 6 by 2 is 12 plus 1 carry gives 13

368421.1 ← carry

Continuing

7368421 → 47368421 → 947368421

Now we have 9 digits of the answer. There are a total of 18 digits (= denominator - numerator i.e 19-1 = 18) in the answer. The last 9 digits can be computed by complementing the lower half (with its complement from nine i.e number + complement = 9):

052631578

947368421

Thus the result is 1/19 = 0.052631578,947368421 repeating.

If you picked up 1/29, you'll have to do it till 28 digits (i.e. 29-1). You'll get the following

1/29 = 03448275862068, 96551724137931

Run this on your calculator and check the result!

I do not like this method a lot as it seems academic. Why would someone want to find all the numbers in the ratio, until he is doing some research. For all practial purposes 4-5 digits of the decimal (from the left) are enough. So the next technique is my favoured method.

METHOD 2

Using Division: The earlier process can also be done using division instead of multiplication.

For A/X9, We divide A by (1+X). Incase of 1/19, we divide 1/(1+1), the answer is 0 (lets say N) with remainder 1 (lets say D)

0.0: D (in this case 1) carries forward and become (D*10 + N) i.e. 10. This is then divided by 2 for N = 5

0.05: Next 05 divided by 2 with answer as 2 and remainder 1.

0.052: Next 12 (1 from earlier remiander and 2 from the answer) is divided by 2 for answer 6

0.0526: and so on.

So the answer of 1/19 = 0.0526.

Consider another example, 1/7, this same as 7/49 which has last digit of the denominator as 9. The previous digit to 9 in 49 is 4. So by the sutra we take one more i.e. 5. So we divide the numerator by 5, that is,

...7/5: 1 R = 2; 21/5: 4 R = 1; 14/5: 2 R = 4, 42/5: 8 R = 2 and so on...so the answer is 1/7 = 0.1428

Now try to convert 18/19 into decimals.

(Numerator 18 to be divided by x+1=2)

...18/2:9 R = 0 ⇒ 9/2:4 R:1 ⇒ 14/2: 7 R = 0 ⇒ 7/2: 3 R = 1 and so on.

So the answer for 18/19 is 0.9473...

Now try 18/29.

Can you do it mentally? The answer is 0.6206...

Note this technique can be used for conversion of vulgar fractions ending in 1, 3, 7 including 9 such as 1/11, 1/21, 1/31 - - -- ,1/13, 1/23, - - - -, 1/7, 1/17, - - - - - by multiplying appropriate factors to the numerator and denominator to make them ed with 9.

1/11 = 9/99 → 0.09090909

1/13 = 3/39 → 0.07692307

Amazed at the speed at which you can calculate these sums MENTALLY!!!

People say there are so many small things that we have to remember, instead of one general method of division or multiplication. My answer is once you get used to it the effort in remembering such an useful method is far surpassed by the ease and speed.

Note: 1/17 = 7/119 → This technique becomes a bit difficult when you try to solve with 1/x9, where x is greater than 9. Will discuss other techniques for these type of big fractions.

Sutra 2: "All from nine and the last from 10" comes handy when we wish to subtract a number such as 578 from a number having zeros equivalent or more than the number of number being subtracted e.g. 1,000. Each figure in 578 is subtracted from nine and the last figure is subtracted from 10, yielding 422. The interesting thing is that this problem can be solved from left to right the same way we write and read text.

This sutra can be extended to solve problems such as 60,000 minus 987. We simply reduce the number left after taking out the 0s to account for the number being subtracted (i.e 60 left after removing the 3 left 0s) by 1 and then apply the sutra, to get the answer 59,013.

The concept gives rise to Sub-Sutra 7: "Whatever the extent of its deficiency, lessen it still further to that very extent; and also set up the square of that deficiency". Let me explain this further:

For instance, in computing 92 we go through the following steps:

Let us take 10 as our base. To get 9 to the base 10 we have to subtract 1 from 10.

Since decrease 9 further by 1 to 8. This is the left side of our answer.

On the right hand side put the square of the deficiency, which is 12. Hence the answer is 81.

Similarly, $8^2 \rightarrow (8-2)*10 + 2^2 \rightarrow 64$, $7^2 = 49$.

For numbers above 10, we take the surplus. For example:

$11^2 = (11+1)*10 + 1^2 = 121$

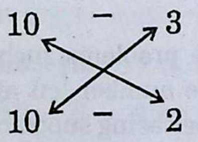
$13^2 = (13+3)*10 + 3^2 = 169$

and so on.

This is based on the fact that $a^2 = (a + b)(a - b) + b^2$ where a is the number whose square is to be found and b is the deficit (or surplus) from nearest factor of 10.

Sutra 3: "Vertically and crosswise" has many uses. One very useful application is helping children who are having trouble with their tables above 5 × 5. For example 7 × 8. 7 is 3 below the base of 10, and 8 is 2 below the base of 10. i.e

7 * 8



$= (10-3) * (10 -2)$

$= 100 - 20 -30 + 6 = 56$

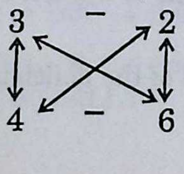
The sutra "vertically and crosswise" is often used in long multiplication. Suppose we wish to multiply 32 by 46.

We multiply "vertically and crosswise":

Vertically 2 × 6=12.

so put down 2 and carry 1.

Then we multiply crosswise and add the two results: $3 \times 6 + 4 \times 2 = 26$,



Add the carried 1 to make it 27

so put down 7 and carry 2.

Finally we multiply vertically 3×4=12 and

add the carried 2 =14.

Result: 1,472.

We can extend this method to deal with long multiplication of numbers of any size. The great advantage of this system is that the answer can be obtained in one line and mentally. With some practice, one can do pretty long multiplications in their heads.

Can you now do some quick calculations for multiplications with 11?

What is 28 * 11? What is 375 *11?

The answer for this is very simple:

28 * 11 = 308

Simply write the 2,(2+8),8 → 2,(10),8 (1 from 10 has to be taken forward and added to 2) so answer is 308

The magic is simplified when you break 11 to (10+1): 28 * 11 = 28 * (10+1) = 280 +28 → 2 (8+2) 8 → 308

375 * 11 = 4125

What is happening here? 375 * 11 → 3,(3+7),(7+5),5 → 3,(10),(12),5 → 4125

This Sutra is used to solve complex non-linear partial differential equations. It deals with calculation of common functions and their series expansions, and the solution of simultaneous, algebraic, transcendental and differential equations.

Multiplication can also be carried out starting from the left, which can be better because we write and pronounce numbers from left to right. Here is an example of doing this in a special method for long multiplication of numbers near a base (10, 100, 1,000 etc.), for example, 96 by 92. 96 is 4 below the base and 92 is 8 below. 100-4= 96 or 100-8=92. So 10000 - 1200 + 32. Applying the Sutra 2 for the calculation 10000 - 1168 = 8832.

Try this 86 * 88 = (90-4) * (90 - 2) = 8100 - 540 +8 (To get to Sutra 2, 100-540+8 is done first) = 8000-432 = (Sutra 2) 7568.

This works equally well for numbers above the base: 105x111=11,655. Here we add the differences. For 205×211=43,255, we double the first part of the answer, because 200 is 2×100.

Sutra 5: If the Samuccaya is the same it is zero.

This formula is useful in solution of several special types of equations that can be solved visually. The word samuccaya has various meanings in different applications. For instance, it may mean a term which occurs as a common factor in all the terms concerned. A simple example is equation "12x + 3x = 4x + 5x". Since "x" occurs as a common factor in all the terms, therefore, x = 0 is a solution. Another meaning may be that samuccaya is a product of independent terms. For instance, in (x + 7) (x + 9) = (x + 3) (x + 21), the samuccaya is 7 × 9 = 3 × 21, therefore, x = 0 is a solution. Another meaning is the sum of the denominators of two fractions having the same numerical numerator, for example: 1/(2x - 1) + 1/(3x - 1) = 0 means we may set the denominators equal to zero, 5x - 2 = 0.

Yet another meaning is "combination" or total. This is commonly used. For instance, if the sum of the numerators and the sum of denominators are the same then that sum is zero. Therefore,

$$\frac{2x+9}{2x+7} = \frac{2x+7}{2x+9}$$

Therefore, 4x + 16 = 0 or x = -4.

This meaning ("total") can also be applied in solving quadratic equations. The total meaning can not only imply sum but also subtraction. For instance when given $N_1/D_1 = N_2/D_2$, if $N_1 + N_2 = D_1 + D_2$ (as shown earlier) then this sum is zero. Mental cross multiplication reveals that the resulting equation is quadratic (the coefficients of x2 are different on the two sides). So, if $N_1 - D_1 = N_2 - D_2$ then that samuccaya is also zero. This yields the other root of a quadratic equation.

Yet interpretation of "total" is applied in multi-term RHS and LHS. For instance, consider

$$\frac{1}{x-7}+\frac{1}{x-9}=\frac{1}{x-6}+\frac{1}{x-10}.$$

Here $D_1 + D_2 = D_3 + D_4 = 2x - 16$. Thus $x = 8$.

There are several other cases where samuccaya can be applied with great versatility. For instance "apparently cubic" or "biquadratic" equations can be easily solved as shown below:

$(x - 3)^3 + (x - 9)^3 = 2(x - 6)^3$.

Note that $x - 3 + x - 9 = 2 (x - 6)$.

Therefore $(x - 6) = 0$ or $x = 6$.

This would not work for the apparently quadratic $(x - 3)^2 + (x - 9)^2 = 2(x - 6)^2$, which has no real or complex solutions.

Consider

$$\frac{(x+3)^3}{(x+5)^3}=\frac{x+1}{x+7}.$$

Observe: $N_1 + D_1 = N_2 + D_2 = 2x + 8$.

Therefore, $x = -4$.

This formula has been extended further.

Sutra 6: If one is in ratio the other is zero is often used to solve simultaneous linear equations which may involve big numbers. But these equations in special cases can be visually solved because of a certain ratio between the coefficients.

Consider the following example:

$6x + 7y = 8$

$19x + 14y = 16$

Here the ratio of coefficients of y is same as that of the constant terms. Therefore, the "other" variable is zero, i.e., $x = 0$. Hence, mentally, the solution of the equations is $x = 0$ and $y = 8/7$

(alternatively:

$19x + 14y = 16$ is equivalent to:

$(19/2)x + 7y = 8$.

Thus it is obvious that x has to be zero, no ratio needed, just divide by 2!

Note that it would not work if both had been "in ratio". For then we have the case of coinciding lines with an infinite number of solutions.:

$6x + 7y = 8$

$12x + 14y = 16$

This formula is easily applicable to more general cases with any number of variables. For instance

$ax + by + cz = a$

$bx + cy + az = b$

$cx + ay + bz = c$

which yields $x = 1$, $y = 0$, $z = 0$.

A corollary says solving "by addition and by subtraction." It is applicable in case of simultaneous linear equations where the x- and y-coefficients are interchanged.

For instance:

$45x - 23y = 113$

$23x - 45y = 91$

By addition:

$68x - 68y = 204 \Rightarrow 68(x - y) = 204 \Rightarrow x - y = 3.$

By subtraction:

$22x + 22y = 22 \Rightarrow 22(x + y) = 22 \Rightarrow x + y = 1.$

Again, by addition, we eliminate the y-terms:

$2x = 4$, so $x = 2$.

Or, by subtraction, we eliminate the x-terms:

$-2y = 2$, and so $y = -1$.

The solution set is $\{2,-1\}$.

7

Is Consciousness a Number?

Introduction

What could be simpler than a number? A child typically learns to count almost as soon as he learns to speak. What could be more essential, more ubiquitous than numbers? We use them constantly, often subconsciously. Yet, as with so many things that are simple, essential, and ubiquitous, not many of us stop to think about what a number really is. Among those that do stop to think about it, there is widespread disagreement. What does the current writer hope to add to this situation? He wishes to describe an answer that, for him at least, works spectacularly well.

This answer makes use of the revival of a remarkably effective and comprehensive science of consciousness, Maharishi Vedic Science. This science is based on the experience of pure consciousness through the Transcendental Meditation program, and the understanding of this experience and its integration into daily life. One branch of Maharishi Vedic Science is Maharishi Vedic Mathematics. This branch describes consciousness and its development from a mathematical point of view. A careful examination of one important concept of Maharishi Vedic Mathematics can achieve a breakthrough understanding of the role of number in mathematics and consciousness. This examination will be one of the primary goals of this paper.

This understanding will also enable us to resolve numerous problems that are currently plaguing the philosophy of mathematics. The primary problem in this area is determining the nature of mathematical objects. Until recently, mathematicians have usually taken the point of view that mathematical objects are transcendental, beyond all objects of the senses. But the current generation of mathematicians usually finds themselves unable to defend this position. Maharishi Vedic Mathematics gives us the tools to properly evaluate the nature of mathematical objects, both theoretically and experientially. We will examine this outcome in detail.

Another problem that Maharishi Vedic Mathematics enables us to solve is one whose existence is not widely appreciated at present. This is the assumption that set theory is a foundational theory of mathematics. We will locate the problem in set theory and then see how Maharishi Vedic Mathematics gives us a way to solve it.

Maharishi Vedic Science

It must be obvious that knowledge is based in consciousness. Our knowledge of anything is based on the degree of development of our consciousness. The past few decades have witnessed the rise of a remarkably effective and comprehensive science of consciousness, Maharishi Vedic Science. This science has its roots in the ancient Vedic culture of India, which enjoyed a remarkably high level of civilization. Only remnants of this civilization are available today in modern India.

The modern revival of Maharishi Vedic Science is based on the experience of pure consciousness through the Transcendental Meditation program, the development of this experience through other techniques that originate in ancient Vedic culture, and the understanding of this experience and its integration into daily life. Over 600 studies have demonstrated the effectiveness of the Transcendental Meditation technique in many areas. Maharishi Vedic Science has been established by a remarkable individual, Maharishi Mahesh Yogi, who started his worldwide Transcendental Meditation Movement in 1957.

Maharishi Vedic Mathematics

One branch of Maharishi Vedic Science is Maharishi Vedic Mathematics. This branch describes consciousness and its development from a mathematical point of view, and prescribes, among other things, the Transcendental Meditation technique as an "enjoyable exercise of Vedic Mathematics" (Maharishi 1996, p. 441).

Unity is an important concept in Maharishi Vedic Mathematics. Unity, in both common usage and among mathematicians, is simply the number one. In Maharishi Vedic Mathematics, Unity rises to the status of self-reference. The following diagrams help us to understand this.

When we perceive a cup, an apple, a child, a sentence, a song, or any such object, we are implicitly counting to One, because we are perceiving one such object. If we perceive more than one, we can consider each one individually. We are therefore characterizing each object by the number one. By counting to one, we are attributing the quality of the number one to an object.

But all aspects of the object can be so attributed. What, then, about the object cannot be attributed to oneness? Clearly none. Therefore, from the viewpoint of complete consideration of everything about the object, the number one contains everything about every object. The status of one rises from a derivative attribute to the source of all objects of consideration. Our appreciation of the number one is then reversed.

Now we will see that Unity is both a number and self-referral consciousness. In order for anything to exist, it must be possible to count it, and the first counting number is One. Then we can use other numbers to measure other aspects of the object. Numbers themselves can be counted. As with anything else that is counted, when we count numbers, we start with the number One. When the number One counts itself, or refers to itself, it thereby assumes a role of self-reference.

In order for Unity to rise to self-reference, it cannot count in an explicit way, because to do so would fill up the universe with nothing but the activity of counting. Therefore, One counting itself in a self-referral way must be on the level of knowingness, consciousness without activity, pure silence.

The Absolute Number is this unmanifest level of number. In this writer's opinion, this is what Maharishi is referring to when he says that the number one, when it is zeroed, becomes the Absolute Number: As any number is the modified state of the number one, Unity; and as Unity is an eternal continuum, the number one circled 1 expresses the eternal continuum of Unity, which can only be expressed in terms of zero-in terms of a circle. (Maharishi 1996, p. 614)

As any number is the expression of the number one, when the number one is zeroed it becomes unmanifest, which means in terms of numbers it becomes the Absolute Number. This means that any number zeroed transforms itself into the Absolute Number-any number zeroed becomes unmanifest. (Maharishi 1996, p. 614)

The Absolute Number is a field of pure self-referral consciousness, which does not enter into the field of distinctions but maintains itself as pure consciousness, without activity or distinctions.

We can deduce, therefore, that Unity is both a number and a field of self-referral consciousness. In the following passage, Maharishi verifies this by essentially equating the Absolute Number with the self-referral nature of Unity.

Vedic Mathematics is the mathematics of the Absolute Number, which is the reality of the absolute field (the field of Unity, the field of self-referral consciousness, the field of Samahita, the field of the Veda). (Maharishi 1996, p. 371)

Solution of Problems in the Foundations of Mathematics

Modern mathematics has drifted away from its ancient foundations in arithmetic and geometry. Almost all modern mathematicians take at least a nominal view that set theory is the foundation of modern mathematics. Yet set theory attracts very little research interest from professional mathematicians, and a problem or advance in set theory is usually viewed as having little or no bearing on the rest of mathematics. Maharishi Vedic Mathematics sheds much light on this peculiar situation. We will undertake a simple examination of set theory to show how.

A simple definition of a set is that it is any well-described collection. Each object that belongs to the collection is called an element of the set. We can speak of the set of all the nations of Europe, or the set of all points on a line. The set of the nations of Europe has a finite number of elements, and so it is called a finite set; while the set of all points on a line is an infinite set, since there are an infinite number of points on a line. A set may have no elements at all; there is only one such set, and it is called the empty set. Sets are denoted with braces: the set containing the numbers 0, 1, and 2 is denoted {0, 1, 2}. The empty set is denoted { }. Sets can contain other sets; thus { {} } is the set containing the empty set, which is a different set from just the empty set by itself.

The set theory that the mathematician uses is called pure set theory. In pure set theory, all the elements of a set are other sets. The construction of all such sets starts with the empty set. This allows only such sets as { }, or { {} }, or

{ {}, {{}} }, or an infinite set that contains only other sets. The whole numbers are defined as pure sets in the following way: the number 0 is defined to be the empty set { }; the number 1 is defined to be the set containing the empty set, i.e. the set containing 0, i.e. { {} }; the number 2 is defined as the set containing 0 and 1, i.e. { {}, {{}} }; 3 is defined as the set containing the 0, 1, and 2; and so forth. All mathematical structures are defined by modern mathematics in this way, starting from the empty set.

Pure set theory has an identifiable starting point in the empty set, and a kind of self-reference in that its sets admit only other sets as elements. Nevertheless, it fails to be a true foundational theory, for several reasons:

1. The inside and outside of a set are two distinct realms, and so the concept of a set is based on the assumption of a fundamental duality.
2. Sets are governed by the axioms of set theory, which are not regarded as themselves being sets. Thus the governance of set theory is based on another assumption of a fundamental duality.
3. Since sets may be infinite but the number of axioms of set theory is finite, set theory attempts to approach the infinite using only finite intelligence-it approaches the infinite from a finite point of view.
4. Set theory cannot fully account for self-reference. When the empty set includes itself, it becomes something different from the empty set. Set theory therefore cannot account for the simple ability of the number One to count itself while remaining One. Likewise, set theory cannot account for the self-referral counting ability of any number.
5. Because set theory cannot account for counting, it fails to account for any of the laws of arithmetic. Since arithmetic is fundamental to any branch of

mathematics, set theory is unable to fully account for the laws of any branch of mathematics, even its own laws.

Maharishi Vedic Mathematics rectifies these shortcomings and thus becomes a much better foundational theory for all of mathematics:

1. Numbers, including the Absolute Number, have no inside or outside. They simply count, and that counting may be purely self-referential. There is no fundamental duality in the structure of a number.
2. The laws which govern the Absolute Number and the relative numbers are included in the structure of the Absolute Number itself.
3. Maharishi Vedic Mathematics approaches the infinite from its own level-infinite intelligence. This infinite level of intelligence is accessible to everyone.
4. The number One does not change when it counts itself. This is an unchanging or absolute quality of Unity, which is due to the Absolute Number's support of every number.
5. The Absolute Number is the common source of all the numbers, and so it encompasses both Unity and diversity. Because all the laws of nature are based on numbers, the Absolute Number is also the home of all the laws of nature.

Set theory does provides a framework for associating any mathematical structure with a set. But a framework is not necessarily a foundation, and an association is not necessarily a definition. Those who claimed that set theory was the foundation of arithmetic never proved it and never could have, because they never had any independent theory of arithmetic. Maharishi Vedic Mathematics rectifies this by bringing to light the self-referral nature of number, both in theory and in practice.

The emphasis of Maharishi Vedic Mathematics on the self-referral nature of number finds confirmation in one of the most famous theorems in the foundations of mathematics, G¨odel's Incompleteness Theorem, often simply called G¨odel's Theorem. The discovery of this theorem in 1930 by Kurt G¨odel (1906-1978) made him one of the most famous mathematicians of the twentieth century. There are many ways to state G¨odel's theorem, some of them very technical, but perhaps one of the easiest is to paraphrase Gensler (1984, p. 1): G¨odel's Theorem states that a complete knowledge of arithmetic cannot be reduced to a finite set of rules. G¨odel proved this theorem about arithmetic by using arithmetic. He used the ability of numbers to count themselves and anything else by:

1. transforming the language of his theorem into numbers;
2. constructing a self-referential statement in this language;
3. using self-referral properties of numbers to show that this statement is true;
4. using the same properties to show that no finite set of rules can prove this statement.

Solution of Problems in Philosophy of Mathematics

Among the Western disciplines, mathematics affords us with probably the clearest example of the transcendental quality of knowledge and intelligence. This position is well known in Western philosophy as Platonism. The recent revival of interest in Western philosophy has sparked a widespread interest in the philosophy of mathematics, among mathematicians and non-mathematicians alike, and with this has come a renewed examination of Platonism.

Unfortunately, even though mathematicians in the past have generally defended Platonism, almost none of the current generation of mathematicians does so. A recent well-known work in the philosophy of mathematics by Hersh (1997) highlights this situation as he throws out Platonism and develops his own theory, called humanism, which takes the view that mathematics is solely a social activity. We will now examine this and other views of Hersh and see how their shortcomings can easily be rectified by Maharishi Vedic Mathematics.

> Platonism . . . says that mathematical entities exist outside space and time, outside thought and matter, in an abstract realm independent of any consciousness, individual or social. (Hersh, 1997, p. 9)
>
> Why do mathematicians believe something so unscientific, so farfetched as an independent immaterial timeless world of mathematical truth? (p. 11)
>
> We don't ask, How does this immaterial realm relate to material reality? How does it make contact with flesh and blood mathematicians? We refuse to face this embarrassment. . . . (p. 11)
>
> To answer, you have to forget Platonism, and look in the sociocultural past and present. . . . (p. 12)
>
> Mathematical objects are created by humans. Not arbitrarily, but from activity with existing mathematical objects, and from the needs of science and daily life. (p. 16)
>
> 1. Mathematics is human. . . .
>
> 2. Mathematical knowledge isn't fallible. . . .
>
> 3. There are different versions of proof and rigor. . . .
>
> 4. Mathematical object are a distinct variety of social-historical objects. (p. 22)

The kind of Platonism that Hersh describes is in some ways like the reality described by Maharishi Vedic Mathematics, except that Hersh's Platonism prescribes no method for contact between the abstract, timeless realm and individual consciousness. Maharishi Vedic Mathematics gives many procedures, especially the Transcendental Meditation technique, for enabling the individual to contact the transcendental realm of pure consciousness and pure orderliness beyond the activity of the senses and the mind. A realm beyond all activity, containing nothing but the structures of pure orderliness, is thereby raised from the shrouds of mysticism to a simple matter of experience. This field is not created by humans, but it can be fully experienced by anyone.

Providing both techniques and proper understanding to support the techniques makes Maharishi Vedic Mathematics itself a science. Since this realm of pure orderliness is also the realm of pure consciousness, the Atma or Self of everyone, it develops individual awareness, and it results in more alertness and increased ability to function optimally in every way. In this way, Maharishi Vedic Mathematics supports all the other sciences.

Another important point in relation to modern mathematics is that the experience of pure consciousness which Maharishi Vedic Mathematics supplies is beyond the intellect. One can say that the experience of pure consciousness is knowledge, but it must then be said that this knowledge is different from intellectual knowledge. The philosophy of Maharishi Vedic Mathematics finds confirmation from Kurt G¨odel, whose philosophical views are much less well known than his famous theorem. Considering his status as a famous research mathematician, they are quite illuminating. Consciousness is connected with one unity. A machine is composed of parts. (Wang 1996, quotation 6.1.21)

It is a primitive idea of our thinking to think of many objects as one object. (Wang 1996, quotation 8.2.1)

A set is a unity of which its elements are the constituents. It is a fundamental property of the mind to comprehend multitudes into unities. (Wang 1996, quotation 8.2.2)

> It is surprising that the fact that multitudes are also unities leads to no contradictions; this is the main fact of mathematics. (Wang 1996, quotation 8.2.2)

> Numbers appear less concrete than sets. They have different representations and are what is common to all representations. For example, we add or multiply by dealing with a collections of two indeterminate things. (Wang 1996, quotation 8.2.5)

Another mathematician who confirms the philosophy of Maharishi Vedic Mathematics is Franklin Merrell-Wolff (1887-1985). He was a professor of mathematics at Stanford. Early in his academic career, in 1916, he left Stanford, spent twenty years in solitude studying Vedic texts, had several realizations of consciousness which he said confirmed certain traditional Vedic teachings (those of Shankara and Advaita Vedanta), and wrote several books describing these realizations and how they fulfilled Western philosophy and science. In the following passage, he describes his experience of Consciousness as Unity, which is essentially different from the duality in which knower and known are separate.

> Once one has had a glimpse of the Consciousness which I have called the High Indifference, it readily becomes clear that there can be but one ultimate Element. Spirit is not divided from matter in any final sense. This is simply the truth concerning the subject and the object, given in other terms. In that State where I realize Myself as identical with all space and with all objects, there is no division between and "I" and a "not-I." This is equivalent to saying there\ is but one Element. But the same thing cannot be said concerning the seeming relationships within the shell of the subject-object consciousness. For here the subject has the appearing of being different from the object, but these two stand in some relationship to each other. (Merrell-Wolff 1994, p. 144)

Exercises of Maharishi Vedic Mathematics

Maharishi Vedic Mathematics is not merely a philosophy. It encompasses a range of practices which develop self-referral consciousness. In the following quotation, Maharishi refers to these as "exercises" and includes among them the Transcendental Meditation and TM-Sidhi programs.

Exercises in Vedic Mathematics lie in fathoming and stabilizing selfreferral consciousness through my Transcendental Meditation and TM-Sidhi Programme, and studying different aspects of the Vedic Literature in the light of my Vedic Science and Technology, in the light of self-referral consciousness, because the study of self-referral consciousness is the study of invincible, perfect defence. (Maharishi 1996, p. 350)

The Transcendental Meditation technique is a simple, natural, effortless technique which takes the mind from the surface level of thinking to the source of thought, pure consciousness, consciousness in its self-referral state, a state of restful alertness which is also the source of all creativity. As the mind settles down during this technique, both mind and body experience deep rest, and stress and fatigue are released. The result of this experience is increased creativity and intelligence, which naturally lead to increased efficiency, enjoyment, and success in all aspects of life.

The marvel of Transcendental Meditation is that it increases orderliness in the individual's awareness while expanding the awareness to encompass all space and time.

Maharishi explains this as follows:

The precision and order in the practice of Transcendental Meditation makes the procedure systematic and mathematical. Transcendental Meditation utilizes the holistic value of Natural Law to materialize an all-directional, progressive and enriching effect in all fields of space and time. It is this theme of Vedic Mathematics available through

Transcendental Meditation that promotes absolute order from the common basis of all activity in the universe-the selfreferral, unified field of consciousness. (Maharishi 1995, p. 173)

Transcendental Meditation is the programme of Vedic Mathematics because it enlivens and utilizes the holistic value of Natural Law to materialize an all-directional effect in all fields of space and time.

It is this theme of Vedic Mathematics available through Transcendental Meditation that promotes absolute order from the common basis of all activity in the universe-the self-referral, unified field of consciousness. (Maharishi 1996, p. 441)

The TM-Sidhi program is an advanced program based on the Transcendental Meditation program. The TM-Sidhi program develops the ability to function directly from the level of pure consciousness. This has the effect of dramatically accelerating the growth of mind-body coordination, by developing the ability to think and act from the level where thought and action are most powerful. The TM-Sidhi program is an application of Yoga, one of the branches of Vedic Literature. Other branches of Vedic Literature contain other programs for developing pure consciousness and gaining the support of the laws of nature. These programs also belong to Maharishi Vedic Mathematics, as Maharishi explains: Exercises in Vedic Mathematics are the programmes to actually gain custody and command over the Laws of Nature-the programmes detailed in all the twenty-seven areas of the Vedic Literature. (Maharishi 1996, p. 348)

Subjective research verifying the approach of Maharishi Vedic Mathematics The regular practice of the Transcendental Meditation and TM-Sidhi programs and other exercises of Maharishi Vedic Mathematics naturally results in higher states of consciousness, beyond the ordinary waking, dreaming, and sleep states of consciousness. The TM technique gives the experience of Transcendental Consciousness, a state of silent restful alertness, a state of pure self-referral without any mental activity. As the practice progresses, the silence of Transcendental Consciousness becomes integrated with the activity of daily life, until they are experienced simultaneously. This is the state of Cosmic Consciousness, where activity is present with and fully supported by the silence of the pure value of the Self, or Atma. We here quote from Maharishi (1976), who quotes experiences of various individuals identified only by their initials and state of residence. The following experience was recorded by S.B. of California:

> Boundless infinitude, beautiful bliss, total silence. In activity a powerful silent wholeness rests on the surface of everything. Abeautiful softness connects and interfuses all I see. (Maharishi 1976, p. 77)

As an individual continues to experience Cosmic Consciousness, he begins to perceive finer values, until he is able to perceive the finest value of creation. This state is known as God Consciousness. The following experience was recorded by D. S. of California:

> I am perceiving finer and finer values of bliss and beauty, and feeling part of a whole beautiful plan. How small the individuality is, yet how powerful with the growing value of all-encompassing wholeness. The body is perceived as a tiny boundary which the Self flows in and out of. (Maharishi 1976, p. 83)

Continued experience of refined perception in God Consciousness ultimately leads to the ability to perceive any object of attention in terms of the unbounded Self. This state is known as Unity Consciousness. The following experience was recorded by M. H. of California:

> The continuum of Self within myself and outside of me just seems to have a very enjoyable, lively, intimate quality-as if my Self is smiling and radiating everywhere. For the

first time I feel concretely, tangibly clearer and closer to all that is meant by God, to all that He is. (Maharishi 1976, p. 85)

The following is another such experience, recorded by C.C. of British Columbia: In the Unity [experience], I would see the object and know it to be myself. A few examples are: I saw the stars, planets, and galaxies of the universe spread out before me and instantly knew them to be part of myself-the same sense of recognition one experiences when one sees unexpectedly one's own image reflected in a mirror.

I saw a series of people's faces . . . and knew intuitively that 'I am all that'-I saw all Creation . . . and knew it to be existing inside myself. (Maharishi 1976, p. 79)

The present writer has had many satisfying experiences with various exercises of Maharishi Vedic Mathematics. One of these exercises is the examination of a body of knowledge in the light of a certain verse of R.

Veda, the first and most important branch of Veda and Vedic Literature in Maharishi Vedic Science. This verse is called the R.

The verses of the Ved exist in the collapse of fullness in the transcendental field, self-referral consciousness, the Self, in which reside all the devas, the impulses of creative intelligence, the laws of nature, responsible for the whole manifest universe.

He whose awareness is not open to this field, what can the verses of the Ved accomplish for him? Those who know this level of reality are established in evenness, wholeness of life. (R. k Veda 1.164.39)

The exercise calls for the analysis of any area of modern knowledge in the light of the eight subsections of this verse. I chose several topics in elementary arithmetic and did this analysis. Almost immediately I was aware that number has the property of self-reference. Shortly thereafter, I was sitting with my eyes closed, enjoying Transcendental Consciousness, when I saw the field of intelligence before me as a glowing plane.

I understood that this field of intelligence is identical with my own self-referral state of consciousness. On this glowing field, a circle of light traced itself out, starting at the bottom, proceeding clockwise, and ending up again at the bottom.

I understood this circle to be a circle of intelligence. As the first half of the circle was tracing itself out on the left side, I perceived that certain points on the left side of the circle were associated with certain areas of mathematics. When the tracing reached the top of the circle, I understood that the left side is what we have known so far, and the right side is what we will know.

I believe that "we" refers to human beings in general. As the tracing proceeded down the right side of the circle, both the circle and the plane became increasingly brighter. When the tracing reached the bottom, the circle and the plane were so bright that the circle became lost on the background of the plane, and the plane merged into my own state of self-referral consciousness.

Objective research verifying the approach of Maharishi Vedic Mathematics Objective scientific research verifies that Maharishi's Transcendental Meditation technique takes one from a state of duality toward unity. For example, a study by Haynes et al. (1976) showed that, on average, the more one experiences Transcendental Consciousness, the more one experiences the following:

- high scores on a standard creativity test;
- high synchrony in brain waves between the left and right hemispheres of the brain; and
- shorter recovery time in the muscles and nervous system after a receiving a disturbing stimulus.

Another study by Orme-Johnson et al. (1981) found that the TM-Sidhi program produces larger increases in brain wave coherence and creativity than the TM program alone. Together, these studies show the TM and TM-Sidhi programs result in the growth of unified values within the individual and integration of mind and body.

A study by Kory and Hufnagel (1974) examined the effect of practicing the Transcendental Meditation technique on high school students at three different high schools. At each school, the students practicing the TM technique were matched against students studying psychology. The study compared grade point average and anxiety using standard tests before and during the experimental period. In two out of three schools, there was a significant reduction in anxiety and a significant increase in grade point average among the TM students, while the psychology students showed no significant changes.

Additional objective verification of the growth of Unity through Maharishi's Transcendental Meditation technique is in the finding of increased self-actualization, which is the use of the full value of one's inner potential, including stability, integration, maturity, warmth, and adaptability. Two studies (Seeman et al. 1972, and Nidich et al. 1973) showed that those who practice the Transcendental Meditation program for only two months scored significantly higher on a standard measure of self-actualization, compared to those who do not practice TM.

Another remarkable phenomenon is that this effect spreads from the individual to the society. Numerous studies have confirmed that a group in which the participants collectively practice the TM-Sidhi Program and the number of participants is at least the square root of one percent of the population of a city, state, or country, has the effect of significantly increasing positive social trends and decreasing negative ones in that area.

For example, Dillbeck et al. (1987) showed that when such groups were present in: Delhi, India; Puerto Rico; Manila, Philippines; and Rhode Island, USA; such measures as crime rate, accident rate, alcohol sales, cigarette sales, unemployment rate, and even pollution level, all decreased significantly.

Another study by Dillbeck (1990) showed that when a long-term group was present in Fairfield, Iowa, supplemented by short-term groups in: Washington, D.C.; the Hague, Netherlands; and Amherst, Massachusetts; then a "violence index" consisting of auto accidents, homicides, and suicides decreased significantly.

Both of these papers ascribe these results to a field effect, a term borrowed from physics to indicate a factor which is present throughout space and time, itself unseen but producing measurable effects.

Summary and Conclusion

Modern society is based on science, and scientists regards mathematics as the ideal science. Virtually everyone studies modern mathematics for eight to twelve years at a young age. Yet this has not been enough to keep modern society from being riddled with problems.

Modern mathematics attempts to approach the infinite from the level of the finite, and other modern disciplines imitate this approach. We have seen that Maharishi Vedic Mathematics allows us to approach the infinite from its own level, and that this approach is the basis of Maharishi Vedic Science.

We have seen that Maharishi Vedic Mathematics is easy, enjoyable, natural, profound, all-encompassing, and useful. We have seen how Maharishi Vedic Mathematics has its roots in the self-referral nature of pure consciousness, which is the source of thought and the source of all creation. We have seen that this field of self-referral consciousness is both perfectly orderly and perfectly awake. We have seen that Maharishi Vedic Mathematics can easily solve problems currently plaguing the foundations and philosophy of mathematics. We have examined the exercises of Maharishi Vedic Mathematics, and we have seen subjective and objective results of these exercises.

It should be obvious that, if there is any truth to all this, Maharishi Vedic Mathematics should be a part of the educational curriculum at all levels as a supplement to modern mathematics. It only remains for everyone to discover for themselves that the science and technology of Maharishi Vedic Mathematics can fill our lives with truth, knowledge, and love.

8

Multi Dimensions of Time and Space

Cubic Model of Time and Space

The cubic model of time and space is E2×E3 space time model. The spatial boundary in the role of dimension manifests real 4-space and the solid domain in the role of dimension manifests real 5-space. This spacetime model works out smooth transition and continuum from real 4-space to real 5-space by attaining transcendence from the manifestation layers. This precisely is being availed in the organizational of Manasara.

The scripture opens with the prayer of Lord Brahma, the overlord of real 4-space and ends with the chiselling of third eye of Lord Shiva, the overlord of real 5-space. Precisely, this enlightenment of the transcendence phenomenon of real 5-space is the subject matter of instructions of Sri Sri Shiva Mahapuran, the text book of real 5-space. Therefore, the cubic model of spacetime is the basic spacetime model for working out the geometric continuum from manifestations layers on the format of real 4-space to the transcendence phenomenon on the format of real 5-space.

Spherical Model of Time and Space

Like cube, sphere is another representative regular body of 3-space. Therefore, parallel to the cubic model of the spacetime, we shall be having spherical model of spacetime. Here as well essentially the Cartesian expression for the spacetime frame would be $E^2 \times E^3$ as the surface boundary of the sphere shall be manifesting hypercube-4 while the solid domain of the sphere would be manifesting hypercube 5.

The domain-boundary ratio of the cube as well as of the sphere being identical, as such both cubical and spherical models can be worked parallelly. However, both these models have their distinct characteristics and hence their distinct utilities and applied values. The glaring difference between the geometric set ups of cube and sphere lies in the set ups of their boundaries.

The boundary of cube is synthetic set up of as many as 6 surface plates while on the other hand the spatial boundary of the sphere is a single integrated sheet. As such the architectural designs of vibrant synthetic reality would depend upon our capacity to exploit the geometric richness of the representative regular bodies of 3-space in the role of dimensions as spacetime formats.

Space, Time and Spacetime of Manasara

For proper appreciation of space, time and spacetime frame of Manasara, we may have a fresh look at the organizational format of the Manasara. The architectural and sculptural achievement of this scripture lies in bridging the deep void between the geometric set up of real 4-space and real 5-space. It really is bound to be a mental holocaust to imagine being on the one bank of this deep void and thinking of reaching the other bank of this deep void.

How to fill the gap in between the two consecutive integral dimensional spaces? Is the question which is going to be the cause for mental holocaust as the logic would fail the bridge the deep void causing holocaust. Mind would feed the meditative brain the supreme order of nature as a working mechanism of the geometric continuum which runs through the deep voids in between the integral dimensional spaces in terms of the spacetime frames with time being one step ahead of the dimensional order of the start with integral space.

In the instant case of the deep void in between real 4 & 5-space we shall be requiring spacetime frame of spatial dimensional order of real 5-space. Linearly, real 4-space is A8 and the solid flux shall be yielding 7 geometries of 3-space of linear dimensional order and hence we shall be having 8 + 7 = 15 linear dimensions which shall be working out 15 = 3 × 5 i.e. 5 solid dimensions for real 5-space. The 7 geometries of 3-space would be parallel to cube devoid of surfaces; cube with 1 surface, cube with 2 surfaces, cube with 3 surfaces and so on cubes with all the 6 surfaces. Therefore, the deep void of real 4 & 5-space would be filled with the help of 2 space sequentially stitching the boundary of 3-space. This sequential stitching mechanism of the boundary of 3-space is the supreme mechanism of nature in terms of which the geometric continuum is being attained by having transition through transcendence. Two space as 0 manifestation layer with 2-space playing the role of domain fold shall be admitting 3-space in the dormant state at its origin in a sealed state. With activisation of the dormant origin, seal of the origin would be unlocked and there would be phenomenal transformation with transcendence from 0 manifestation layer to 1 manifestation layer. This transcendence phenomenon starting with the origin of 2-space to the origin of 3-space is the life span. It is this span which works out the age of everything.

Multi-dimensions of Time and Space

The multi-dimensions of time and space as timespace frame for bridging the deep void in between two integral dimensional spaces and consequential attainment of geometric continuum for the vibrant synthetic reality of transcendence phenomenon through manifestation layers is the basic subject of interest for the scientists and technologists.

When like the mechanism of bridging of the deep void between real 4 & 5-space, the deep voids between real 5 & 6 spaces and between real 6 & 7 space stand worked out, it may be possible for us to have the bliss of Vedic comprehension of the Bindu Sarowar as the constituent of 6-space, Vishnu-lok, a point of the manifestation layer with 4-space in the role of dimensions, 5-space in the role of boundary, 6-space in the role of domain and 7-space in the role of unity state.

The blissful state is the unity state of consciousness which can be attended by all of us the Vedic Way through Transcendental meditation, a phenomenon of 5 space format, parallel to the transcendence phenomenon which would fill the deep voids between the different states of consciousness as well as between the different manifestation layers.

Sum Up

Knowledge and Organisation Knowledge

We can appreciate the distinction of knowledge and organization of knowledge in term of an affine space and metric affine space. In general terms, we may approach this distinction in terms of space and dimensional space. In more concrete terms, we can approach it in the context of our physical world in terms of 'volume' as attribute of 3-space content and the volume organized as a 'cube'. The organization of a cube requires points, lines and surface plates in addition of the volume, lines and surface plates. All these set ups of points, edges, surfaces and volume are to be made out of the available content. The unique mechanism of transformation of an affine space into metric affine space or of space into dimensional space or of 3-space or space into dimensional space or of 3-space content as volume into 3-space lump as cube has real mathematics which deserves to be learned.

Great Vedic comprehension of this mechanism as that the vibrant synthetic reality accepts synthetic monads, is responsible for manifestation layers with sealed origin, (n-2) space playing the role of dimensional of n-space and the domain-boundary ratio of hypercube-n comes to be $A^n:2nA^{n-1}$. This makes the dimensional order of space of two prongs distinct for odd and even dimensional spaces as much as that odd (2n-1) dimensional space shall be having linear dimensional order as $(2n + 1) \times (2n - 3) \times \ldots\ldots.. \times 3 \times 1$ and for even (2n) dimensional space to have linear dimensional order as $2n \times (2n-2) \times (2n-4) \times \ldots\ldots \times 6 \times 4 \times 2$.

Further the domains boundaries sequence shall be yielding sequential ratios for hypercube-n as A^n: $2nA^{n-1}$: $2n \times 2(n-1) \times A^{n-2}$: $2n \times 2\,(n-1) \times 2\,(n-2) \times A^{n-3}$ .... (till the term A0). This sequential order with the above linear dimensional order for dimensional space and the concept of synthetic monad of n parts making each part as 1/n, when worked together would yield the answer for the above great mechanism of nature, which in the language of Ramayana would mean a difference of two Anguls (measuring units) between the arm of Lord Rama and Kag-Bhushandi and in the language of Shri Bhagwad Mahapuran that every string used by Mother Yashoda to bind Lord Krishna turned out to two Anguls short.

In mathematical language, it means that the representative regular bodies of dimensional spaces i.e. interval, square, cube and hypercubes shall be working out a linear sequential order which shall be transforming respective affine spaces into respective metric spaces or spaces as dimensional space or space contents (hyper volumes) as hypercubes which can be expressed as $Bn = (A + 2)^n$ which means the hypervolume Bn remains only as $(B-2)^n$ and the remaining content stands consumed by the boundary for wrapping the domain $(B-2)^n$.

Now if we work out the above $B^n = (A + 2)^n$ equation for n= 1, 2, 3, 4, and so on, we shall be convincing ourselves about the above mechanism of the nature. For n=1, we shall be getting $B^1 = A^1 + 2A^0$. This straight way follows from the domain-boundary formulation for 1-space body. For n=2, we shall be getting $B^2 = A^2 + 4A^1 + 4A^0$. Here the first pair of terms follow straight from the domain-boundary formulation of 2-space body. The second term 4A1 would yield 8A0 boundary components. But here we shall be reaching boundary of the boundary which means, we shall be taking a synthetic step of two parts to reach the boundary. This will make each part being of ½ value. Applying this, we shall be getting boundary of boundary as $½ (8A^0) = 4A^0$. Likewise, for n=3, we shall be getting $B^3 = A^3 + 6A^2 + 12A^1 + 8A^0$. Here as well the first pair of terms follows straight way from the domain-boundary formulation of 3-space body. The third term which shall be taking us to the boundary of the boundary would accordingly can be computed as $½ (4 \times 6A^{2-1}) = 12A^1$. By the same logic we shall be reaching the next term as $1/3 (2 \times 12A^{1-1})$ for n = 4, 5, 6 and so on.

Now let us appreciate the above mathematics in terms of the geometric set ups of close interval, square and cube and we shall be comprehending that the equation Bn = $(A+2)^n$ for n=1 yields the geometric set up of close interval as length (A^1) with two end points ($2A^0$), for n=2 yields the geometric set up square as area (A^2) with four boundary lines ($4A^1$) and four end points ($4A^0$) and for n=3 yields the geometric set up of cube as volume (A^3) with six surface plates ($6A^2$), 12 edges ($12A^1$) and 8 corner points ($8A^0$).

This will help us appreciate the greatness of Vedic model. Further this will also help us appreciate how Bn remains $(B-2)^n$ as domain in the synthetic set up and the remaining content is consumed by the boundary of $(B-2)^n$ as domain and this would be a never ending sequence which impliedly means, we shall be exhausting the contents and even logically would be demanding the existence in a void, the impossibility, which is the rule of ad-infinitum reduction and this will prove the Fermat's last theorem as that $z^n = X^n + y^n$ and $(z-2)^n = (x-n)^n + (y-2)^n$ can not hold simultaneously. Accordingly, every Vedic scriptural organizational has a great mathematics in store for us.

Lord Shiva, Sun, Man, Atom and Manasara

Lord Vishnu, is the Lord of real 6-space. The seat of Lord Vishnu is the heat of Lord Shiva. 6-space is located at the organ of 5-space. Dwadash Adityas (12 suns) are the 12 boundary components of hypercube-6. As many as 12 hypercube-5 are required to constitute the boundary of hypercube-6. Lord Shiva meditates upon His Lord Vishnu and multiplies 12 folds into Vishnu-lok. Isha punished enlightens us that Purusha, in the sun is Lord Vishnu. Ayurveda enlightens us that Pursuha is constituted by 6 Dhatus and Atman is the 6th Dhathu. First five Dhatus are Panchmahabhut. Lord Shiva is the Lord of Panchmahabhut. Lord Vishnu is the Atman. The studies of the set up of the atom also leads us to the conclusion that electronic configuration around the nucleus i.e the micro world accepts for format of hypercube-4 while nucleus itself concepts for format of hypercube 4 while nucleus of the nucleus would accept the format of hyupercube-6, Mansara accepts Lord Vishnu, the overlord of 6-space as the Lord of the measuring rod. Lord Brahma, the four head lord is the Lord of measure. Manasara begins with the prayer of Lord Brahma and ends with the chiselling of the eye of the Lord. As such the conclusion comes to be that Manasara adopts the orgnizational format of hypercube 5 at the boundary of hypercube-6 which would be parallel to the worlds of Lord Shiva, Aditya, Panchmahabhut and nucleus of the atom.

Spacetime Fram of Mansara

The spacetime frame of Manasara as such is the spacetime frame of hypercube-4 transcending into hypercube-5. This is a transition through transcendence from the spatial dimensional order into solid dimensional order. As such practically this would amount to acceptance of cube or sphere as representative bodies of 3-space in the role of dimensions. The spatial boundary of the cube shall be creating 4-space while the solid volume shall be creating 5-space. This really is very exciting role of vibrant synthetic world being tamed by the Manasara.

Three Space in the Role of Dimension

To have proper comprehension and appreciation of the organizational set up of Manasara we have to learn the mathematics & geometry of 3-space in the role of dimension. With 3-space as dimension we shall be creating 5-space domain with 4-space in the role of boundary and 6-space in the role of origin. This is the Panchmahabhut world. We may compare it with our present day thought mathematics, science and technology which accepts physical world as a 3-space domain.

Cube and sphere are the representative regular bodies of 3-space. They are miniatured replicas of linear manifestation layers. With sealed origin and un-manifest dimensions, the other two folds namely, spatial boundary and solid domain when come in action would create transition through transcendence from hypercube-4 to hypercube-5. To have proper appreciation of this phenomenon, we may have a fresh look at the geometric set up of the cube by viewing it as a cube of 10 units length, 10 units breadth and 10 units height.

We shall be having 1000 unit cubes synthesized as a single cube of 1000 units volume. If we want to know that how many edges / pillars we shall be having of this set up of 1000 cubes, then the answer of Vedic mathematics would be simple in terms of Devnagri numerals 3 and 6 being reflection images of each other like reflection pair 01 and 10. The tri-monad i.e. the monad of three parts shall be yielding 363 and then the length being 10 units so we shall be having the figure 3630. Let us count for ourselves and verify this answer.

Now this reflection property of the light and the visible light spectrum of 7 colours on the one hand and the need of only seven edges to sequentially connect all the 8 edges on the other hand and the property that real 4-space in linearly A8 would help us appreciate the interconnection of the artifices of whole numbers 3 & 7 in the context of 3-space. In this background, we shall be having a bliss of the first quarter of fist Mantra of the Atharvved.

Yeh Trishapta Prayanti Vishwa

Trishapta (3&7) wraps Vishwa (this world)

The message is that 3-space at the boundary of 4-space is a geometric set up of 3 linear dimensions which can take us sequentially uptill 7 linear steps.

The domain and boundary are distinct set ups. If domain represents and becomes the Vedic domain, the boundary would represent and would becomes the Upved. Sthaptyaved being Upved of Athavved and Manasara being the scripture of Sthaptyaupved, therefore, we shall be requiring comprehension of the Vishwa as is the enlightenment of the first quarter of the first Mantra of the Atharvved to appreciate the organization of Mansara.

Therefore the conclusion of all conclusions for us is that for proper learning of Vedic science & technology, there is a need for redrawing the school mathematics syllabi.

Om Tat Sat Iti

The sole syllable Om is the essence of pure knowledge and it is the truth and the ultimate.

Fermat's Last Theorem

Fermat's Last Theorem (FLT) has remained most famous unsolved mathematical problem. This theorem has also attracted the attention of Dr.S.K.Kapoor. He has published his results about the truth of this property of numbers. He has worked out the theorem with different approaches. Essentially all these approaches are geometric in nature but still these are different in characteristics and as such these may be taken as different proofs of the theorem. His approaches and conclusions are not only having intrinsic academic values but also these have historic angle as much as that these accept inspiration as well as working operations from a distant source i.e. Vedic literature. These results are also published in point of time than the proof now accepted in the academic circles.

Three of these proofs of FLT were published in "Modern Science & Vedic Science" (Volume 3 No.1, 1989) with the caption "Vedic Mathematical Concepts and Their Applications to Unsolved Mathematical Problems: Three Proofs of Fermat's Last Theorem". Chapter-10 of the book "Vedic Geometry" (1994) as well takes up some of the aspects of this property of numbers under the caption "Chapter 10: Conclusions and their Applications to the Solution of Fermat's Last Theorem". Then followed the book "Fermat's Last Theorem and Higher Spaces Reality Course" (1996) in which in addition to the outline of different formats of the approaches to the proof and in addition to the general proof of FLT, the result of the theorem has been generalised as Generalised FLT for whole range of hypercubes.

The concepts of power sets and the application of different place value systems to test the truth of any given triple of whole numbers have also been introduced which may prove to be very handy for tests of the truth of this property known as FLT. These proofs are on different formats which are worked out as:

1. On Domain Format
2. On Simplex Format
3. On Values Square Format

4. On the Format of Domain As Dimension Of Another Domain
5. On the Format of Hypercubes

Fermat's Last Theorem speaks out the property of linear dimensional order only. The cause of the restriction being the linear dimensional order so we get the restriction of powers being three and higher. The results generalized the theorem as a geometric property of hypercubes by a shift from linear dimensional order to a spatial dimensional order. The generalized statement comes to be as that "no hypercube can be duplicated" because of the interlocking of (n-2) space with n-space as dimension and domain respectively.

A Simple Proof of Fermat's Last Theorem

It is a shame that Andrew Wiles spent so many of the prime years of his life following such a difficult path to proving Fermat's Last Theorem, when there exists a much shorter and easier proof. Indeed, this concise, elegant alternative, reproduced below, is almost certainly the one that Fermat himself referred to in the margin of his copy* of Bachet's translation of Diophantus's Arithmetica--how arrogant of future chroniclers of mathematical history to insinuate that the garlicky Gallic genius was incapable of formulating the crucial insight! Why, a child can follow the logic below...however, for the benefit of those readers who are no longer children, a short set of notes fleshing out the argument is provided after the proof.

The Theorem: $x^a + y^a = z^a$ has no positive integer solutions (x, y, z, a) for a > 2. (Pierre De Fermat, 1601-1665)

The Proof:

(I) At least one of the following two sentences is true.

(II) The preceding sentence is false.

(III) $x^a + y^a = z^a$ has no positive integer solutions (x, y, z, a) for a > 2.

Fermat's Proofs of Special Cases

One of the methods used also by mathematicians in their struggle with hard problems is "Divide and Conquer" - splitting the problem into sub-problems that are easier to handle. This tactic appears in connection with FLT already in Fermat's writings, which contain a kernel of a proof for the special case n=4. This is the only proof in number theory that Fermat left (it had space in the margin of Diophantus' book), and it uses the infinite descent method. Fermat also dealt with the special case n=3, as we learn from his letter to Christian Hugeness, but, habitually, he did not leave a proof for this case.

The proof for the special case n=3 was found one hundred years later by Leonhard Euler, the greatest mathematician of the eighteenth century. In number theory, Euler is the follower of Fermat: many theorems stated by Fermat were proven by Euler. In 1753 Euler wrote to Christian Goldbach about the proof, but he did not publish it. In 1770, in his book Anleitung zur Algebra (A Guide to Algebra), Euler gave a proof for this case. This proof was slightly different from the one he gave to Goldbach, and in it Euler used algebraic numbers, which are created by adjoining numbers of the form $\sqrt{3}$, for example, to the group of whole numbers, creating numbers of the form $a+b\sqrt{3}$, where a and b are integers. Although these numbers are not integers, Euler found them useful to solve Diophantine equations.

Euler proofs use the infinite descent method, strengthening Fermat's claim that this method "will lead to marvellous advancement in the theory of numbers". Except the proof technique, the proof for the case n=3 is different from the one for the case n=4, and this explains why, in his letter to Goldbach, Euler said that he did not see any hope to generalize those two special proofs into a general proof. Desperately, Euler asked to make a thorough search in Fermat's home, in hope to find a lead to the missing proof.

The mail connection between Euler and Goldbach lasted 35 years, and is an important source of information about Euler's work. In the beginning of this correspondence, in 1729, Goldbach motivated Euler to deal with Fermat's work. In a letter from 1748, Euler mentioned for the first time Fermat's Last Theorem ("a very nice theorem", he wrote later). Goldbach's Conjecture appeared in a letter from 1742. This conjecture is the main direct contribution of Goldbach to the development of mathematics. His indirect influence, as a friend to Euler, is much more important. In 1843, sixty years after Euler death, his grand grandson published the full correspondence.

The proof for the case n=5, in a way that was an enlargement of the way Euler used for his proof of the case n=3, was created by two partners: the German Peter Gustav Lejeune Dirichlet, the pupil and friend of Gauss, who was 20-year-old, and the French Adrien Marie Legendre, one of the leaders of the French Academy of Sciences and follower of Fermat and Euler in building the foundations of number theory, who was 73-year-old. Dirichlet divided the proof needed for this case into two cases, and in July 1825 he presented to the Academy of Sciences a proof to one of those cases. Two months later, Legendre published, as an appendix to his successful book Théorie des nombres (Number Theory), a proof for the second case, completing Dirichlet's proof. Two months later, Dirichlet supplied a nicer proof for the second case, a kind of natural continuation of his proof for the first case.

Fourteen years later, Gabriel Lamé proved the theorem for the case n=7. Why did Lamé skip the proof for n=6? If the numbers A, B and C are a solution to Fermat's equation for the power of 6, namely $A^6 + B^6 = C^6$, then the numbers A^2, B^2 and C^2 are a solution to the power of 3, contradictory to the proof that FLT is true for this power. Generally, if we had proven FLT to any power k, than the theorem is valid to all the multiples of k. The reason for this is that if the numbers A, B and C are a solution for the power mk, then the numbers A^m, B^m and C^m are a solution to the power of k, contradictory to the proof that FLT is true for this power. This proof technique is called proof by contradiction (or Reductio ad absurdum, if you prefer to say it in Latin). Fermat's infinite descent method is a special type of this technique.

It follows from this proof that it is enough to prove FLT just for powers that are primes greater than 2, since each other power is a multiple of these primes (except 4, but for this power the theorem has already been proven by Fermat). It is a big step toward reduction of the scope of the problem, but it is still leaving us with the need to prove FLT for an infinite set of powers. An elegant proof of the infinitude of primes appears in Euclid's book, the Elements.

In the first two centuries that passed since Fermat wrote his theorem, number theory had developed enormously. After the pioneer work of Fermat, Euler and Legendre, in 1801 Carl Friedrich Gauss published his book, Disquisitiones arithmeticae (Arithmetic Research), which turned number theory from a collection of isolated results into a systematic mathematical theory. Gauss, one of the greatest mathematicians of all times, called number theory "The Queen of Mathematics". Gauss was very pedantic about the quality of works he gave for publication, and many of his achievements were published many years after they were accomplished. A proof by Gauss for the case n=3 of FLT, different from that of Euler but also using the infinite descent method, was published after his death.

Scriptures Formats

Knowledge and organization of knowledge are two different aspects of pure knowledge. One of the beautiful features of Vedic scriptures is that their knowledge is organized on geometric formats of real four and higher dimensional spaces. Here these are being reproduced in the form and sequence these have been taken up during this interaction:

Scriptures Format 1: Om

Om is one syllable scripture complete in itself. The past, present and future and all that transcends three folds of time is verily Om. This way the sole syllable expresses as four folds scripture complete in itself exhaustively covering the past, present and future and all that transcends the three folds of time on tetra-monad format.

Scriptures Format 2: Ved

Ved means knowledge; the whole range of knowledge. This the whole range of knowledge as one Ved was reorganised by Brahmaa-Rishi Ved Vyas into four Veds: Rig Ved, Yajur Ved, Sam Ved and Atharv Ved. Literally Vyas means diameter of a circle (a representative regular body of 2-space which in the role of dimension manifests as 4-space.

The organization of one Ved as four Veds is parallel to the one syllable Om unfolding on tetra-monad format i.e. as an expression of four folds. With this shift to a diameter, there is a shift from a monad (linear order) to di-monad (spatial order). This gives expression for 1 as 01 whose reflection member is 10 i.e. ten.

The above shift from 1 to 10 in that order and sequence, gives us:

(1) 10 + 01 + 10 = 21, the total number of branches of Rig Ved.

(2) 10 × 10 + 01 × 01 = 101, the total branches of Yajur Ved

(3) 10 × 10 × 10 = 1000, the total branches of Sam Ved

(4) 10 -01 = 9, the total branches of Atharv Ved

Scripture Formats 3: Idol of Lord Brahma

Lord Brahma, the creator the supreme, is the four heads Lord, with each head equipped with two eyes. He sits majestically on lotus seat of eight petals and meditates in cavity of his heart on his Lord, Lord Shiva and multiplies himself as ten Brahmas.

The idol of Lord Brahma as of four heads Lord, with two eyes in each head, sitting on lotus seat of eight petals and meditating upon his Lord, Lord Shiva, in the cavity of his heart, as such, accepts the format of hypercube-4 whose four folds are 2-space in the role of dimension (parallel to two eyes in each head), 3-space in the role of boundary (parallel to eight components of boundary of hypercube-4 and eight petals of the lotus seat of Lord Brahma), 4-space in the role of domain (parallel to the domain of four heads Lord) and 5-space in the role of origin (parallel to the seat of Lord Shiva, five heads Lord, in the cavity of the heart of Lord Brahma).

Scripture Formats 4: Idol of Lord Shiva

Lord Brahma, the creator the supreme, is the four heads Lord, with each head equipped with two eyes. He sits majestically on lotus seat of eight petals and meditates in cavity of his heart on his Lord, Lord Shiva and multiplies himself as ten Brahmas.

The idol of Lord Brahma as of four heads Lord, with two eyes in each head, sitting on lotus seat of eight petals and meditating upon his Lord, Lord Shiva, in the cavity of his heart, as such, accepts the format of hypercube-4 whose four folds are 2-space in the role of dimension (parallel to two eyes in each head), 3-space in the role of boundary (parallel to eight components of boundary of hypercube-4 and eight petals of the lotus seat of Lord Brahma), 4-space in the role of domain (parallel to the domain of four heads Lord) and 5-space in the role of origin (parallel to the seat of Lord

Shiva, five heads Lord, in the cavity of the heart of Lord Brahma).

Scripture Formats 5: Incarnations of Lord Vishnu

Lord Sakand is incarnation of Lord Vishnu. Lord Sakand a six heads lord with four eyes in each head and twelve long arms, is the incarnation of Lord Vishnu. Lord Sakand establishes the transcendental worlds by meditating upon eternity, eternally sustained by Dhruva (pole star).

This description is parallel to the description of hypercube-6 with 4-space (parallel to four eyes in each head) in the role of dimension, 5-space in the role of boundary, 6-space in the role of domain and 7-space in the role of origin. The other incarnations of Lord Vishnu accepts this format.

Scripture Formats 6: Measure and Measuring-Rod

Lord Vishnu is the god of measuring-rod and Lord Brahma is the god of measure.

The Lord Vishnu is the overlord of real 6-space and Lord Brahma is the overlord of real 4-space.

Real 4-space plays the role of dimension of real 6-space. As such while the measuring-rod is constituted by domain (6-space), the measure remains at the dimension (4-space).

Scripture Formats 7: Shad-chakra Format of Human Body

Vedic (Upanishadic) enlightenment is that human body admits Shad (six) Chakras (circuits) format. The external characteristics of the Shad-Chakras are 2, 4, 6, 8, 10 & 12.

The external characteristics of Shad-Chakras of human body are parallel to the number of boundary components of hypercubes-1, 2, 3, 4, 5 & 6.

This format of Shad-Chakras is the format of the order of 6-space whose presiding deity is Lord Vishnu. The creator-the-supreme, Lord Brahma, overlord of real 4-space as such is the presiding deity of the measure of the human body.

Scripture Formats 8: Organization of Knowledge of Sri Vishnu Puran

Sri Vishnu Puran is a scripture of 6 Ansha (parts). The organization of knowledge of all the six Ansha of Sri Vishnu Puran is of 126 chapters.

This organization of knowledge is as per the measuring-rod and the Shad-Chakra format of human body as sustained within the Vishnu-lok (6-space domain)

On the artifices of whole numbers this organization admits expression as:

$6 \times (1 + 2 + 3 + 4 + 5 + 6) = 6 \times 21 = 126.$

Scripture Formats 9: Vishwa-Rupa

Shatpath Brahman preserves the enlightenment : Tawastha had one son. He had three heads and six eyes. And three mouths. So was his form (Rupa). That is why he was named - Vishwa Rupa (form of the world).

This is parallel to the world of three dimensions and domain-boundary ratio of the representative regular body of 3-space being A3:6A2. The enlightenment preserve in the Shatpath Brahman as such is the enlightenment about the geometric setup of the representative regular body of 3-space. Being a representative regular body so is the representative form of the dimensional space / world.

Scripture Formats 10: Trishapta (3 and 7)

"Yeh Trishapta Paryani Vishwa" (This world is enveloped by Trishapata i.e. 3 and 7). Parallel to it is that 3-space has 7 geometries of signatures (0, 1, 2, 3, 4, 5, 6) corresponding to the cube with no surface plate, cube with 1 surface plate, cube with 2, 3, 4, 5 & 6 surface plates respectively.

Scripture Formats 11: One, Two, Three and Eight

Swateshawara Upanishad preserves the enlightenment that sequential order 1, 2, 3 and 8 leads to Moksha (liberation).

Parallel to it is that on the artifices of whole numbers, the sequence 1, 2, 3 and 8 gives us 1×1, 1×2, 1×3 and 2×4. Parallel to it on geometric formats the message is that the order of linear dimensions works uptil first three dimensions and a step ahead is the world of spatial dimensions. In other words while 3-space admits linear dimensions, the 4-space admits spatial dimensions.

Scripture Formats 12: 03 to 13

The Sushmana-Nari is the central nerve which runs through and coordinates the Shad Chakra of human body accepts the measuring-rod constituted by hypercubes-1 to 6. It means that starting with the close interval and reaching uptil hypercube-6 is the range which constitutes the measuring-rod for the Sushmana-Nari running through the Shad Chakras.

The hypercube-1 (close interval) as representative regular body of 1-space accepts three versions of interval parallel to three geometries of 1-space. The hypercube-6 as representative regular body of 6-space accepts 13 versions parallel to the 13 geometries of 6-space.

Lord Vishnu is the presiding deity of the measuring-rod being the overlord of 6-space. Srimad Bhagwad Gita is the divine song of Lord Krishna, the incarnation of Lord Vishnu.

Each of the chapters of Srimad Bhagwad Gita is precisely accepting the measuring-rod with Lord Vishnu as its presiding deity and as such each chapter opens with 3 words title and ends with 13 syllables Pushapika (colophon) parallel to the three geometries of 1-space to 13 geometries of 6-space.

Scripture Formats 13: 120 Years Cycle

The Jyotish Shastra accepts 120 years cycle for the planets' effects upon human life. This is the geometric organization of 4-space in the role of dimension creating 6-space as domain with complete coverage for the boundary of this domain with 120 hypercubes-4.

This is there as A6:12A5 is the domain-boundary ratio of hypercube-6 and further B5:10C4 is the domain-boundary ratio for each of the boundary components. With this, the 12 boundary components of hypercube-6 get enveloped within 12 × 10 = 120 hypercubes-4.

Alternatively, 4-space itself being a spatial dimensional set up, it gives us arrangement for 120 = 60 + 60 and further 60 = 2 + 8 + 18 + 32. It is parallel to the capacity of the orbits of an atom to retain electrons. As such, on either side of the nucleus, the set up admits expression as 2 + 8 + 18 + 32 + 32 + 18 + 8 + 2 = 120.

Scripture Formats 14: 13-Edged Cube is Hypercube-4

Within hypercube-4, the 12-edged cube has one degree freedom of motion along the fourth dimension. As such 12-edged cube gets fixed in 4-space as 13-edged cube. The 13th edge of the cube is supplied by the fourth dimension. Because of the spatial dimensional character of the dimension of 4-space, the 13th edge permits motions of a cube with a spatial pivot. The motion of the cube as a whole in reference to such pivot is the unique cosmic arrangements like that of pole star. The Pushpikas (colophons) beneath all the 18 chapters of Srimad Bhagwad Gita enlighten us about the 6×3=18 different ways of outward motions along all the three dimensions from all the six surfaces of the cube.

Scripture Formats 15: Things Transform Just With the Attention at the Origin-I

Things transform just with the attention at the origin. Let us have attention at the centre of cube / origin of 3-space and everything starts transforming; the cube splits into 8 sub-cubes and 3-space splits into 8 octaves. The origin accepts 8 sub-cubes / 8 octaves enveloping. The 4-space flourishes from within at the seat of origin and everything transforms from 3-space to 4-space.

Scripture Formats 16: Things Transform Just With the Attention at the Origin-II

Things transform just with the attention at the origin. Let us have attention at the centre of hypercube-4 / origin of 4-space and everything starts transforming; the hypercube-4 / 4-space splits into 10 components. The origin accepts 10 components boundary enveloping. The 5-space flourishes from within at the seat of origin and everything transforms from 4-space to 5-space.

The scripture (Yog Vashishta) preserves the enlightenment as that Lord Brahma (the overlord of 4-space) when meditates in the cavity of his heart upon his lord, Lord Shiva, Lord Brahma multiplies himself into 10 Brahmas.

Scripture Formats 17: Things Transform Just With the Attention at the Origin-III

With just attention at the origin of 4-space/hypercube-4, the things transform into that of 5-space / hypercube-5 enveloped within ten 4-space / 10 hypercubes-5. As from spatial dimension things transform into solid dimensional order and as 3-space / cube admits split up as 8 octaves / 8 sub-cubes. Therefore, just with the attention at the origin we see simultaneous happening of split up into 10 components boundary and within it as within 8 components boundary. This is parallel to the simultaneous organization of Rig Ved Samhita as a scripture of 10 Mandals and also as of 8 Ashtaks. With this, with just attention, our mundane world set up of 3-space starts transforming as transcendental Vedic world of the order of Rig Ved.

Scripture Formats 18: Things Transform Just With the Attention at the Origin-IV:

Simultaneous Internal and External Expansions: With just attention at the origin of a space, say n-space / hypercube-n, the interactive process gets initiated and there follows simultaneous internal and external transformations and expansions to the next higher space set up.

If we just have attention at the set up of a cube, the outer surfaces, as 2×3 boundary components / surface plates, being spatial in organization, as to be set to fly off in the outer space and to dimensionalize it as 4-space. Two of the boundary components together to constitute a di-monad format and the remaining four of them to constitute four spatial dimensions for 4-space. This external transformation and expansion as dimensionalize 4-space is parallel to the internal transformation and expansion at the seat of origin as 4-space. This way there is parallel internal and external emergence of 4-space.

Likewise, if we have attention at the set up of a hypercube-4, the outer solids, as 2×4 boundary components / solid components, being solid in organization, as to be set to fly off in the outer space and to dimensionalize it as 5-space. Three of the boundary components together to constitute a tri-monad format and the remaining five of them to constitute five solid dimensions for 5-space. This external transformation and expansion as dimensionalize 5-space is parallel to the internal transformation and expansion at the seat of origin as 5-space. This way there is parallel internal and external emergence of 5-space.

Further, if we have attention at the set up of a hypercube-5, the outer hypersolids, as 2×5 boundary components/ hypersolid components, being hypersolid in organization, as to be set to fly off in the outer space and to dimensionalize it as 6-space. Four of the boundary components together to constitute a tetra-monad format and the remaining six of them to constitute six hypersolid dimensions for 6-space. This external transformation and expansion as dimensionalize 6-space is parallel to the internal transformation and expansion at the seat of origin as 6-space. This way there is parallel internal and external emergence of 6-space.

In general, if we have attention at the set up of a hypercube-n, the outer hypersolids, as 2×n boundary components / hypersolid components, being hypersolid in organization, as to be set to fly off in the outer space and to dimensionalize it as (n+1) space. (n-1) of the boundary components together to constitute a {(n-1)-order} monad format and the remaining n+1 of them to constitute n+1 hypersolid dimensions for (n+1) space. This external transformation and expansion as dimensionalize (n+1) space is parallel to the internal transformation and expansion at the seat of origin as (n+1) space. This way there is parallel internal and external emergence of (n+1) space.

Scripture Formats 19: Things Transform Just With the Attention at the Origin-V

Compactification at the Origin: With just attention at the origin sequentially the simultaneous inward and outward expansion would unfold the compactification of whole range of dimensional spaces at the origin.

The attention at the origin of 3-space would initiate the interaction process of the mind and everything would start transforming and there would emerge simultaneous, inward and outward expansion of 4-space. With this, 3-space / cube as such shall be having placement at the origin and there emerging outward and inward, and as such all around, 4-space in its display. It may be expressible as 4, 3, 4. Three space at the centre and 4-space on its either side. The further attention would transform the very interactive process as much as that while on the one hand the attention would get centred at the origin of 4-space which would expand it outward as well as inward as 5-space and on the other hand the previous stage expression of 3-space at the centre and 4-space around would set into initiation for the 3-space with available degree of freedom of motion to play the role of dimension and as such there would be dimensionalization of 5-space. This simultaneous unfoldment, transformation and expansion as {(4, 3, 4) and (4, 5, 4)} and in general {(n, n-1, n) and (n, n+1, n)} is uniquely there because of the compactification at the origin of whole range of real spaces. (The spaces where n-space plays the role of dimension of n+2 space).

Scripture Formats 20: Things Transform Just With the Attention at the Origin-VI

Continuity Maintained by Bridging of Gaps: The Sushmana-Nari is the central nerve which runs through and coordinates the Shad Chakra of human body accepts the measuring-rod constituted by hypercubes-1 to 6. It means that starting with the close interval and reaching uptil hypercube-6 is the range which constitutes the measuring-rod for the Sushmana-Nari running through the Shad Chakras.

The hypercube-1 (close interval) as representative regular body of 1-space accepts three versions of interval parallel to three geometries of 1-space. The hypercube-6 as representative regular body of 6-space accepts 13 versions parallel to the 13 geometries of 6-space.

Lord Vishnu is the presiding deity of the measuring-rod being the overlord of 6-space. Srimad Bhagwad Gita is the divine song of Lord Krishna, the incarnation of Lord Vishnu. Each of the chapters of Srimad Bhagwad Gita is precisely accepting the measuring-rod with Lord Vishnu as its presiding deity and as such each chapter opens with 3 words title and ends with 13 syllables Pushapika (colophon) parallel to the three geometries of 1-space to 13 geometries of 6-space.

Scripture Formats 21: Vykata, Avyakata, Avyakato-avakatat and Purusha #1

Srimad Bhagwad Gita preserves the enlightenment of the order of the Reality as Vyakta, Avyakata, Avyakto, Avyakato-avyakatat and Purusha.

Vyakta means expressed. It is the expressed world, Vishwa, 3-space.

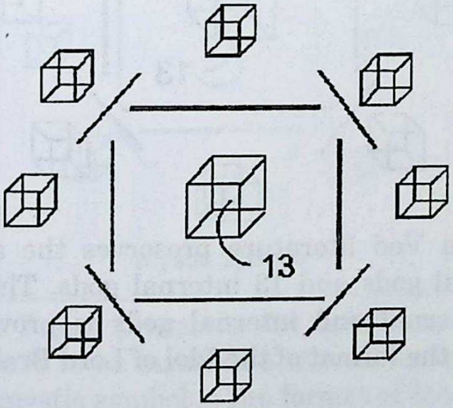
Avyakata means unexpressed which is yet to manifest in the Vishwa, 4-space world.

Avyakato-avyakatat means which is at the base of Avyakata. It is the transcendental base, a 5-space world.

Purusha is fully expressed transcendental consciousness within human frame. It is the consciousness world of Atman (soul) domain, Vishnu-lok, the Purusha in the Surya (sun), the 6-space reality.

Scripture Formats 22: Vykata, Avyakata, Avyakato-avakatat and Purusha #2

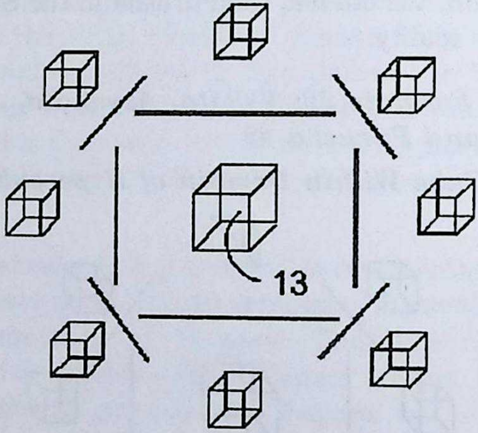
13-Edges Cube Within Domain of Hypercube-4



The above format gives us 8 solid components of the boundary of hypercube-4 and the 12-edged cube gets fixed along the degree of freedom available to it within domain of hypercube-4 and as such manifest as 13-edged cube. There being a degree of freedom of motion along each of the four dimension of 4-space, as such there are 4 different ways for fixation of 13-edged cube within the domain of hypercube-4 and hence within 4-space. Otherwise as the 4-space itself is a spatial dimensional space, as such we get first format for the Great debate of enlightenment and knowledge within the kingdom of King Janak where everybody except Ashtavakra were held up at two fixations for the artifice of whole number 13 while Ashta Vakra won the debate by completing all the four fixations for the artifice of whole number 13.

Scripture Formats 24: Vykata, Avyakata, Avyakato-avakatat and Purusha #4

32 External Gods and 13 Internal Gods:



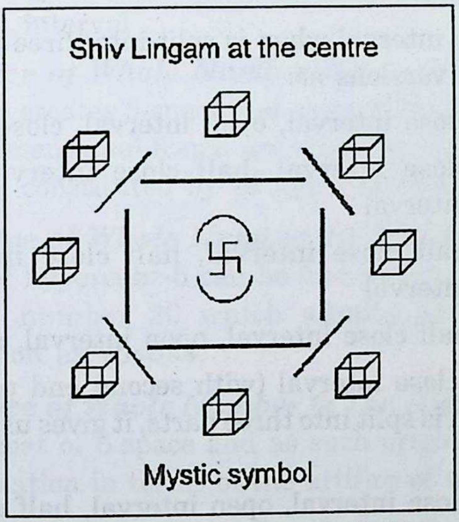
Sthapatya Ved literature preserves the arrangement of 32 external gods and 13 internal gods. The format for these 45 external and internal gods is provided by the hypercube-4, the format of the Idol of Lord Brahma, creator the supreme.

$A^4 : 8B^3$ gives us that within 4-space, the 8 boundary components would get fixed with 8×4=32 coordinates. This together with the fixed 13-edged cube within the domain would provide an arrangement of 32 + 13 = 45 fixations.

The above organization of the format of hypercube-4 with the availability of 45 characteristics arrangements would help us comprehend the enlightenment of Ur-Mahabharatam about the text of Srimad Bhagwad Gita as that this text is of 745 Shalokas out of which 700 Shalokas are the expressed Shalokas while 45 are the unexpressed Shalokas. The available text of Srimad Bhagwad Gita with us is the text of 700 Shalokas and the remaining 45 unexpresed Shalokas remain unmanifest in this world and these are lively and available at the format beneath.

Scripture Formats 25: Vykata, Avyakata, Avyakato-avakatat and Purusha #5:

32 Installation of Shiva Lingam at the Centre of Mystic Symbol



Sri Shiva Puran preserves the enlightenment as that Shiva Lingam is to be installed at the centre of the mystic symbol. The mystic symbol is the format of the Idol of Lord Brahma, the creator the supreme. Shiva Lingam is the solid dimensional expression of the Shiva Lok (5-space). The centre of the mystic symbol is the centre of hypercube-4 / origin of 4-space. The origin of 4-space is the seat of compactification of dimensional spaces from 5-space onwards. The Shiva Lingam emerges from within the origin.

Scripture Formats 26: Description of the Wheel of Cause Brahman #1

Sawteshwara Upanishad describes the wheel of cause Brahman as of 1 (Nemi), 3 (Vrit), 16 (Antah), 50 (Aras), 20 (Pratya), 6 (Ashtak), 1 (Pash, Vishwarupa), 3 (Marg), 2 (Paap, Punya) and 1 (Karan). The artifices of whole numbers availed in that sequence are: 1, 3, 16, 50, 20, 6, 1, 3, 2, 1.

Artifices of Whole Numbers 1 & 3: One as 1-space in the role of dimension creates 3-space.

Artifice of Whole Number 16: The tri-monad, a monad of 3 parts, and an interval of 3 parts gives us 16 versions as:

(1) close interval when is split into three parts, it gives us 4 versions as:

- (a) close interval, open interval, close interval
- (b) close interval, half close interval, half close interval
- (c) half close interval, half close interval, close interval
- (d) half close interval, open interval, close interval

(2) half-close interval (with second end point missing) when is split into three parts, it gives us four versions as:

- (a) close interval, open interval, half close interval
- (b) half close interval, half close interval, half close interval
- (c) half close interval, close interval, open interval
- (d) close interval, half close interval, open interval

(3) half-close interval (with first end point missing) when is split into three parts, it gives us four versions as:

(a) half close interval, open interval, close interval

(b) half close interval, half close interval, open interval

(c) open interval, close interval, half close interval

(d) open interval, half close interval, close interval

(4) open interval when is split into three parts, it gives us four versions as:

(a) half close interval, open interval, half close interval

(b) half close interval, half close interval, open interval

(c) open interval, close interval, open interval

(d) open interval, half close interval, half close interval

Artifice of Whole Number 50: 3-space in the role of dimension creates 5-space and as such 10×5=50 coordinates of solid dimensional frame are required to fix the boundary of 5-space constituted by 10 hypercubes-4.

Artifice of Whole Number 20: The domain-boundary fixation of hypercube-5 can be had in terms of the artifice of whole number 20 which admits arrangement and organization as 20=5×4.

Artifice of Whole Number 6: The origin of hypercube-5 is the seat of 6-space and as such origin of hypercube-5 admits fixation in terms of the artifice of whole number 6.

Artifice of whole number 1, 3, 2 & 1: The origin of origin of hypercube-5 is the seat of 7-space. The artifices of whole numbers 1, 3, 2 & 1 together are the arrangement of the artifice of whole number 7. Going from the artifice of 1 to the artifice of 3 is the going from dimension to domain and going from 1 to 2 is going from the dimension to the boundary.

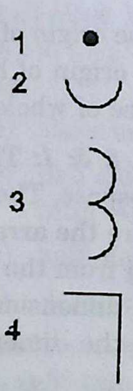
The above expression of the wheel of "cause Brahman" takes us uptil the origin of the origin of the transcendental world of 5-space.

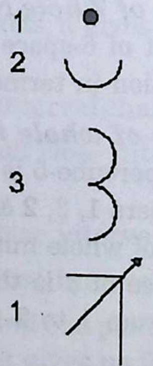
Scripture Formats 27: Description of the Wheel of Cause Brahman #2

Sawteshwara Upanishad describes the wheel of cause Brahman as of 1 (Nemi), 3 (Vrit), 16 (Antah), 50 (Aras), 20 (Pratya), 6 (Ashtak), 1 (Pash, Vishwarupa), 3 (Marg), 2 (Paap, Punya) and 1 (Karan). The artifices of whole numbers availed in that sequence are: 1, 3, 16, 50, 20, 6, 1, 3, 2, 1. In the reverse order the artifices of whole numbers give us the sequence as 1, 2, 3, 1, 6, 20, 50, 16, 3, 1. This sequence admits expression as: 1, 2, 3, 4=1, 4 leads to 6, 6 leads to 9+7+3+1, 4*5=20 leads to 5-space with 4-space boundary and 5*(1+2+3+4)=50 solid points fixation of boundary, the tri-monad format of solids gives us 6-space at the centre of 5-space as 5+6+5=16 and 3 as tri-monad format and 1 as the linear order of 3-space.

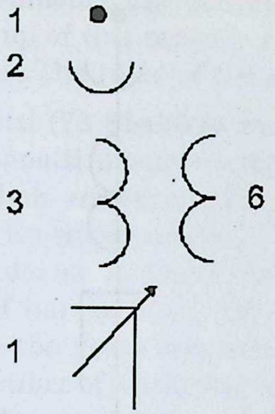
On Om formulation, which is the basic formulation for organization on 4-space format, sequentially the above artifices of whole numbers leads to the following phases of organization of the wheel of Cause Brahman.

The phase of artifices of whole numbers 1, 2, 3, 4 and 1, 2, 3, 1 :

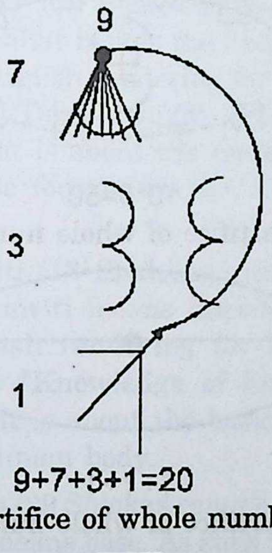




The phase of artifice of whole numbers 6:

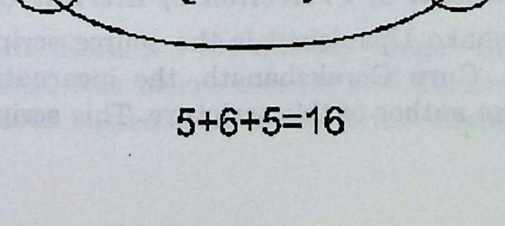


The phase of artifice of whole number 20:



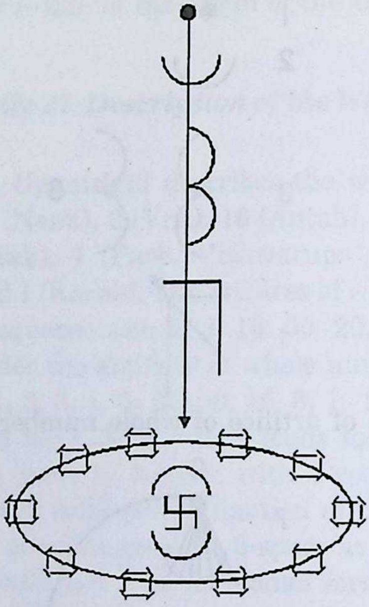
9+7+3+1=20

The phase of artifice of whole number 16:



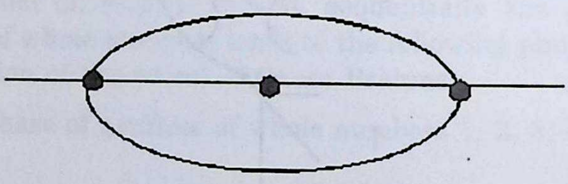
5+6+5=16

The phase of artifice of whole number 50:



10*5=50

The phase of artifice of whole number 3:



1+1+1

The phase of artifice of whole number 1:

1

Scripture Formats 28: Gorakshako Upanishad (Enlightenment of Protection of Eternal Glow) #1

Gorakshako Upanishad is the source scripture of the Nath sect. Guru Gorakshanath, the incarnation of Lord Shiva is the author of this scripture. This scripture is also known as Sidha Sidhanta Padhati. This is a scripture of 6 Upadeshas (enlightenment). The scripture range is of 353 Shalokas. The split up of this range is as of 73, 39, 13, 29, 84 and 115 Shalokas. The titles of the six Upadeshas are:

- Pinda Utapatti (73 Shalokas range): Pinda means body and Utapatti means emergence. As such the simple English rendering for the title of first Upadesha is "Emergence of Body". The enlightenment of this Upadesha is the enlightenment is the emergence of human body. This enlightenment is organised on the geometric format of 73 Shalokas range. The artifice of whole number 73 has been well utilised for the organisation of this Upadesha.
- Pinda Vichaar (39 Shalokas range): Pinda means body and Vichhar means the deep thought. As such the simple English rendering for the title of second Upadesha is "Deep thought about the body". This enlightenment is about the basic characteristics of the geometric format for the life flow within the human body.
- Pinda Samviti (13 Shalokas range): Pinda means body and Samviti means knowledge. As such the simple English rendering for the title of third Upadesha is "Knowledge of human body". This enlightenment is about the basic knowledge of life within the human body.
- Pinda Adhara (29 Shlokas range): Pinda means body and Ahdara means base. As such the simple English rendering for the title of fourth Upadesha is "The ultimate base of human body". This enlightenment is about the ultimate base of life within the human body.
- Samras Karanam (84 Shlokas range): Sam means common, Ras means essence and Karanam means ultimate sustenance. As such the simple English

rendering for the title of fifth Upadesha is "Common essence of the ultimate sustenance of life". This enlightenment is about the common essence of the ultimate sustenance of life within body.

- Avadhut Yogi Lakshanam (115 Shalokas range): Avadhut Yogi means fully elightened being and Lakshanam means characteristic features. As such the simple English rendering for the title of sixth Upadesha is "The characteristics features of fully enlightened being". This enlightenment takes us to the common ultimate state of enlightenment for the living beings.

Scripture Formats 29: Gorakshako Upanishad (Enlightenment of Protection of Eternal Glow) #2

Guru Gorakhnath, the author of the Upanishad is incarnation of Lord Shiva, the overlord of real 5-space. Therefore, the basic format for the organisation of the scripture gets fixed as of hypercube-5 / real 5-space.

Gorakshako Upanishadic text is of 353 Shalokas range. The artifice of whole number 353 is unique in many ways and it, in the light of the enlightenment of the scripture and the organisation format beneath, helps us reach at the basic geometric lock of manifestation of human body.

It is like the emergence of real 5-space at the origin of affine 3-space. The origin of affine 3-space is the seat of real 4-space and as such the transcendence and emergence at this seat is there because of real 5-space being at the base of this seat. The real 5-space itself being a solid dimensional space, this (solid dimensional order) as such locks the seat and the emerging geometric continuum locks the existence phenomena.

This phenomenon of locking and unlocking at the base of the origin of affine 3-space is the basic phenomenon whose comprehension and chasing is to provide us the enlightenment of the manifestation of human body and transcendence therefrom. This life phenomenon is the phenomenon of eternal glow whose protection within the human body is the subject matter of enlightenment of this Upanishad. One name for this discipline may be the discipline of geometry of Vedic genetic engineering.

Scripture Formats 30: Gorakshako Upanishad (Enlightenment of Protection of Eternal Glow) #3

Guru Gorakhnath (Gorakshnath), the author of the Upanishad, is the incarnation of Lord Shiva and with it the basic format for organisation of the scripture is settled as hypercube-5 / real 5-space.

Om Namah Shivai is the basic Mantra of enlightenment of Lord Shiva. Om is sole-syllable, Namah is a pair of syllables and Shivai is a composition of three syllables. With this, this is the enlightenment of artifices of whole number 1, 2 & 3 and parallel to it 1-space, 2-space and 3-space and finally a tri-monad format and so the solid dimensional order takes us to real 5-space.

The artifices of whole numbers 1, 2 & 3 together also give us $1 + 2 + 3 = 1 \times 2 \times 3 = 6$. Simultaneously, we have 6 non-negative geometries of 5-space. This as such helps us to reach at the organisation of the scripture as 6 Upadeshas parallel to the artifice of 6 supplied by the basic Mantra as well as the available 6 non-negative geometries of 5-space.

Scripture Formats 31: Gorakshako Upanishad (Enlightenment of Protection of Eternal Glow) #4

The enlightenment of Gorakshako Upanishad is the enlightenment of Shiva-lok and is the glow of Sri Shivpuran.

Sri Shivpuran preserves the enlightenment as that the gods who were proud of their powers were made to realise the limits of their powers by Lord Shiva as that they were even not capable of making a split of even the blade of grass.

This is precisely the Shloka 26 of Chapter 16 of Satrudra Samhita and it reads as:

> *"If you are proud and think that you know your strength to be great, O gods, split this blade of grass placed by me, with your own weapons."*

The geometric and algebraic message here is that when we creates structures with the help of a binary composition then the elements of the set at first operation give us the composition elements of the form c = a × b. Then when further we operate such composite elements and the resultant elements $d = c^1 \times c^2$ then d takes us uptil the boundary of a 3-space. The domain of 3-space remains untouched though wrapped within the composed elements.

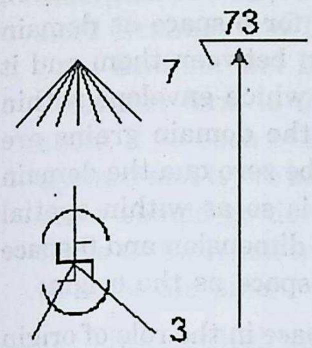
It is because of it that the blade of grass as a solid remains un-split.

This enlightenment and message deserves to be comprehended well to appreciate as to why our set theory approach is creating mathematical knots and while we get dodged and feel as if we can have space filling curves and why the situations like that of everywhere continuous but nowhere differentiable functions and also the situations like the hypercubes-8 onwards creating a decreasing value sequence at all arise for us and we are not in a position even to logically catch as to why this all is happening. This impossibility to split a blade of grass is the proposition and situation at the base of Fermat's Last Theorem and similar like mathematical problems of the day.

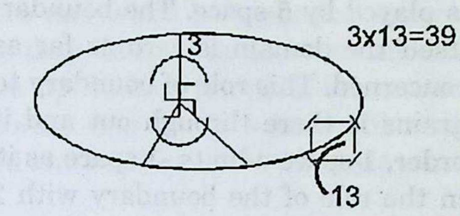
Scripture Formats 32: Gorakshako Upanishad (Enlightenment of Protection of Eternal Glow) #5

The organisation of the Upanishad as 6 Upadeshas admits arrangement for its 353 Shalokas as 73, 39, 13, 29, 84 and 115. This arrangement is there because of the following format beneath because of the geometries of 0, 1, 2, 3, 4 & 5. Here is drawn the arrangement:

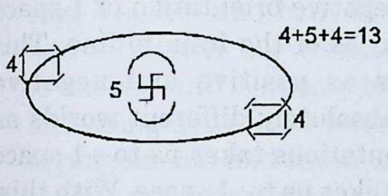
Upadesha-1



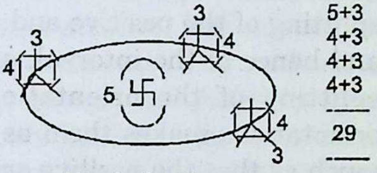
Upadesha-2



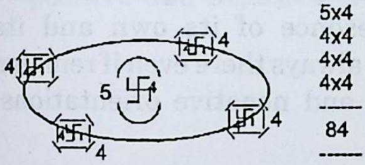
Upadesha-3



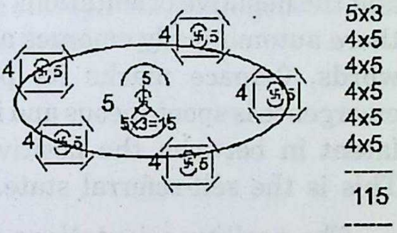
Upadesha-4


3
3
4
4
5
4
3
5+3
4+3
4+3
4+3
29

Upadesha-5


5x4
4x4
4x4
4x4
84
4
4
5
4
4

Upadesha-6


5x3
4x5
4x5
4x5
4x5
4x5
115
4
5
4
5
5
5
5x3=15
4
5
4
5
4
5

Scripture Formats 33: Gorakshako Upanishad (Enlightenment of Protection of Eternal Glow) #6 :

Self-referral State

Sri Shivpuran Samhita preserves the enlightenment that when Lord Brahma (the presiding deity of 4-space) and Lord Vishnu (the presiding deity of 6-space) are face to face and are interacting to know the Brahman, there emerges Lord Shiva (the presiding deity of 5-space) in between them as an infinite pillar of fire (Agani Tatva; the third basic element with Prithvi (earth) as first and Jal (water) as the second).

The geometric message of this enlightenment is that when we are in a Creator's space (the spatial order space), 4-space plays the role of dimension for 6-space as domain and the place for the boundary is in between them and it is played by 5-space. The boundary which envelops within itself the domain is zero as far as the domain grains are concerned. This role of boundary to be zero qua the domain grains is there through out and it is so as within spatial order, 1-space admits -1 space as its dimension and 0-space in the role of the boundary with 2-space as the origin.

This organisation because of 2-space in the role of origin gives us the arrangement because of which there emerges splitting of the positive and negative orientation of 1-space and hence of the interval, and as of the infinite line. This splitting of the orientations as positive and negative orientations makes them as absolutely different worlds as much as that the positive orientations takes us to +1 space and the negative orientations takes us to -1 space. With this, there automatically emerges a jump over a 0-space. In other words, 0-space marks its presence of its own and its emergence is spontaneous and is always there even if remains latent in between the positive and negative orientations. This is the self-referral state.

The positive orientations as 1-space and further in the role of dimension of 3-space gives us as many as 7 geometries for 3-space. Likewise the change of orientation would take us from -1 space to -3 space but in a reverse order as much as that -1 space would become the domain and -33 space as the dimension. The 7 geometries here for -3 space and then the continuity from -3 space to +3 space with 0-space in between -1 space and +1 space gives us two parallel set ups like the pair of objects with their reflection images and because of it a self-referral state makes it possible to have self-evaluation and self-appraisal of one's knowledge and it is there at the format of the division of the brain within human heads as of left and right parts.

The enlightenment of Sri Shivpuran from the stage of emergence of Lord Shiva as infinite pillar of fire in between Lord Brahma and Lord Vishnu goes further as that both Lord Brahma and Lord Vishnu interacted and asked as to what this form is there and when both showed ignorance about it to each other they decided as that one should go upward and try to know uptil which the pillar takes and the other should go downward and try to know uptil what it takes to. Both had tried and realized that it was not possible to scale the pillar.

This enlightenment made them realized the transcendental state of Lord Shiva (the presiding deity of real 5-space) and both of them attained self-referral state and ultimately the transcendental state and realized that it was not sufficient only to go upward and the going upward and going downward were two different organizations and arrangements and those can be known only with reference to the orientation free state of 0-space emerging between the positive and negative orientations spaces.

9

Goldbach's Conjecture

Precisely the conjecture is that every even greater than 2 can be written as a pair of sum of primes. It is part of the famous letter (June 7, 1942) of Christian Goldbach (1690-1764) to great Swiss mathematician Leonhard Euler. Since then this conjecture has remained a brain teaser and unsolved problem of the order of Fermat's Last Theorem and Riemann Hypothesis.

The book "Goldbach Theorem" has four chapters. The proof except Step 2A has been settled in Chapter 1 of the book. All what is presumed as background is only the mature comprehension of the whole numbers. As every even number by definition admits two parts, therefore, it is this property of evens which has come very handy to express even E as M+M. With this the maximum possible pairs with numbers 0 to E have been formed putting a restriction that some of the numbers of the pairs should always be equal to E. The pairs are designated as duplexes. Then the recursive subsets of the source set of maximum number of duplexes of given even number E are constructed and the subset of duplexes with both numbers of the duplexes being primes is reached at. The cardinality of this set is computed as $\geq 1/4 \times \sqrt{E}$.

As $1/4 \times \sqrt{E}$ for $E \geq 64$ is ≥ 2, so after accounting for the duplex (1,E-1), there always remains a minimum of one such duplex whose both members are to be primes and with it the conjecture stands satisfied. Otherwise, this in fact, amounts to extension of the conjecture for $E \geq 64$ from expectation of one partition for E as sum of primes to that of the minimum of $1/4\times \sqrt{E}$ number of partitions for E as sum of primes.

Chapter 2 of the book takes us to Vedic geometric inspiration and approach of di-monad format (a format which accepts entity as of two parts). The interesting property of the geometric setup of "square" of area E is that such a square would be admitting $\sqrt{E}$ as a side of the square. This gives us the ratio of area E and the sum of its four sides as $1/4\times \sqrt{E}$. The conceptual parallelism is much inspiring. This makes Vedic geometric approach as much potentialised approach.

Chapter 3 of the book introduces a slide rule for finding out solutions for E=P+Q. This together with the computer testing (Chapter 4) for the number of solutions of equation E=P+Q adds empirical value to the proof.

Several Proofs of the Twin Primes and Goldbach Conjectures

Proof of Goldbach's Conjecture, the Prime Number Theorem, and Euclid's Logic Provide Proofs of the Twin Primes Conjecture. Proof of the Twin Primes Conjecture Provides Proof of Goldbach's Conjecture.

THEOREM

There are infinitely many twin primes.

Proof of the Twin Primes Conjecture Using Proofs of Goldbach's Conjecture or Using the Prime Number Theorem

The twin primes conjecture (TPC) suggests that there is an infinite number of primes a and b with a difference 2, i.e., $a - b = 2$. Goldbach's conjecture (GC) suggests that every even number greater than 2 is the sum s of two prime numbers a and b, i.e., $a + b = s$ where s is even >2. GC is proved by the author herein below and elsewhere.

For prime numbers a,b,c

. . . . . $a - b = (a + c) - (b + c)$ . . . . even integer ...(1)

and thus, generally,

. . . . . $a - b = 2k$ . . . . k = integer ≥ 1 ...(2)

and since $a + b$ is an even number

. . . . . $a + b = 2n$ . . . . n = integer>2 ...(3)

Now, using (2) and (3) results in

. . . . . $a = n + k$ . . . and . . . $b = n - k$ . . . $n>k$ ...(4)

which say that for every single value of k primes a and b are separated by an interval 2k and occur as numbers n + k and n - k. Suppose that $n_1, n_2, n_3, \ldots, n_r$ are all the twin prime (k=1) intervals. Since number n is unlimited, intervals $n_1, n_2, n_3, \ldots, n_r$ are not all the twin prime intervals, i.e., $n \to \infty$ in $n_1, n_2, n_3, \ldots, n_r$.

Because all numbers n + k and n - k are not prime numbers, equations (2)-(4) are necessary for proving the infinitude but are not sufficient for finding the existence of prime numbers at these locations. However, from the Prime Number Theorem (PNT), I compute that the probability of finding twin primes a at n + 1 and b at n -1 is about 2/logn (k=1) and the number of twin primes in the interval n is about $2n/(\log n)^2$. Also, in contrast to the foregoing actual proof using Goldbach's conjecture, the PNT provides a proof of the TPC as $n \to \infty$. We can also say that,

- the difference of two prime numbers a and b is an even number 2k; and
- the probability of finding primes a at n + k and b at n - k decreases as n increases.

Proof of the Twin Primes Conjecture Using Euclid's Logic

Another actual proof of the TPC is suggested by Euclid's proof that there are infinitely many primes.

Suppose that $i_1 = i_2 = i_3 = \ldots = i_r = 2$ are all the twin prime intervals. Let $I_r = 2_r + 1$, $i_{r+1} = 2$ and $I_{r+1} = 2^{r+1} + 1$ from which it follows that $2I_r = I_{r+1} + 1$. For all values of r, since interval i_{r+1} divides $2I_r$ (without dividing I_r and I_{r+1},) it is yet another interval and intervals $i_1 = i_2 = i_3 = \ldots = i_r$ are not all the twin prime intervals, i.e., in $i_1 = i_2 = i_3 = \ldots = i_r = 2$.

Proof of Goldbach's Conjecture Using Proof of the Twin Primes Conjecture

I can now prove Goldbach's Conjecture by reversing my foregoing Proof of the Twin Primes Conjecture Using Proof of Goldbach's Conjecture. With reference to equations (2)-(4), suppose that not every sum of two primes a and b is an even number $2n>4$. However, since the number of twin primes is unlimited, their sum is an even number 2n unlimited and every sum of primes is an even number.

Mathematics and the Spiritual Dimension

Unfortunately, I think I went too far to the other side. I threw reason to the wind, so to speak, and unceremoniously became a self-ordained "spiritual person." Science, the foundation of which is mathematics, as I saw it, had nothing to offer. It was only years later, when the cloud of my sentimentalism was dissipated by the sun of my soul's integrity, that I was able to separate myself from yet another delusion-the first being the advice of my father, and the second being the idea that I could wish myself into a more profound understanding of the nature of reality.

Math cannot take the mystery out of life without doing away with life itself, for it is life's mystery, its unpredictability-the fact that it is dynamic, not static-that makes it alive and worth living. We may theoretically explain away God, but in so doing we only choose to delude ourselves; I = everything is just bad arithmetic. However, before we can connect with our heart of hearts, our real spiritual essence, we cannot cast reason aside. With the help of the discriminating faculty we can know at least what transcendence is not. Withdrawing our heart from that is a good beginning for a spiritual life.

Mathematics has only recently risen to attempt to usurp the throne of Godhead. Ironically, it originally came into use in human society within the context of spiritual pursuit. Spiritually advanced cultures were not ignorant of the principles of mathematics, but they saw no necessity to explore those principles beyond that which was helpful in the advancement of God realization. Intoxicated by the gross power inherent in mathematical principles, later civilizations, succumbing to the all-inviting arms of illusion, employed these principles and further explored them in an attempt to conquer nature. The folly of this, as demonstrated in modern society today, points to the fact that "wisdom" is more than the exercise of intelligence. Modern man's worship of intelligence blinds him from the obvious: the superiority of love over reason.

Archimedes and Pythagoras

A common belief among ancient cultures was that the laws of numbers have not only a practical meaning, but also a mystical or religious one. This belief was prevalent amongst the Pythagoreans. Prior to 500 B.C.E., Pythagoras, the great Greek pioneer in the teaching of mathematics, formed an exclusive club of young men to whom he imparted his superior mathematical knowledge. Each member was required to take an oath never to reveal this knowledge to an outsider. Pythagoras acquired many faithful disciples to whom he preached about the immortality of the soul and insisted on a life of renunciation. At the heart of the Pythagorean world view was a unity of religious principles and mathematical propositions.

In the third century B.C. another great Greek mathematician, Archimedes, contributed considerably to the field of mathematics. A quote attributed to Archimedes reads, "There are things which seem incredible to most men who have not studied mathematics." Yet according to Plutarch, Archimedes considered "mechanical work and every art concerned with the necessities of life an ignoble and inferior form of labour, and therefore exerted his best efforts only in seeking knowledge of those things in which the good and the beautiful were not mixed with the necessary." As did Plato, Archimedes scorned practical mathematics, although he became very expert at it.

The Abacus: A Mechanical Counting Device

The Greeks, however, encountered a major problem. The Greek alphabet, which had proved so useful in so many ways, proved to be a great hindrance in the art of calculating. Although Greek astronomers and astrologers used a sexagesimal place notation and a zero, the advantages of this usage were not fully appreciated and did not spread beyond their calculations. The Egyptians had no difficulty in representing large numbers, but the absence of any place value for their symbols so complicated their system that, for example, 23 symbols were needed to represent the number 986. Even the Romans, who succeeded the Greeks as masters of the Mediterranean world, and who are known as a nation of conquerors, could not conquer the art of calculating. This was a chore left to an abacus worked by a slave. No real progress in the art of calculating nor in science was made until help came from the East.

Shulba Sutra

In the valley of the Indus River of India, the world's oldest civilization had developed its own system of mathematics. The Vedic Shulba Sutras (fifth to eighth century B.C.), meaning "codes of the rope," show that the earliest geometrical and mathematical investigations among the Indians arose from certain requirements of their religious rituals. When the poetic vision of the Vedic seers was externalized in symbols, rituals requiring altars and precise measurement became manifest, providing a means to the attainment of the unmanifest world of consciousness. "Shulba Sutras" is the name given to those portions or supplements of the Kalpasutras, which deal with the measurement and construction of the different altars or arenas for religious rites. The word shulba refers to the ropes used to make these measurements. Math cannot take the mystery out of life without doing away with life itself, for it is life's mystery, its unpredictability - the fact that it is dynamic, not static- that makes it alive and worth living.

Although Vedic mathematicians are known primarily for their computational genius in arithmetic and algebra, the basis and inspiration for the whole of Indian mathematics is geometry. Evidence of geometrical drawing instruments from as early as 2500 B.C.E. has been found in the Indus Valley. The beginnings of algebra can be traced to the constructional geometry of the Vedic priests, which are preserved in the Shulba Sutras. Exact measurements, orientations, and different geometrical shapes for the altars and arenas used for the religious functions (yajnas), which occupy an important part of the Vedic religious culture, are described in the Shulba Sutras. Many of these calculations employ the geometrical formula known as the Pythagorean theorem.

This theorem (c. 540 B.C.), equating the square of the hypotenuse of a right angle triangle with the sum of the squares of the other two sides, was utilized in the earliest Shulba Sutra (the Baudhayana) prior to the eighth century B.C. Thus, widespread use of this famous mathematical theorem in India several centuries before its being popularized by Pythagoras has been documented. The exact wording of the theorem as presented in the Sulba Sutras is: "The diagonal chord of the rectangle makes both the squares that the horizontal and vertical sides make separately."

The proof of this fundamentally important theorem is well known from Euclid's time until the present for its excessively tedious and cumbersome nature; yet the Vedas present five different extremely simple proofs for this theorem. One historian, Needham, has stated, "Future research on the history of science and technology in Asia will in fact reveal that the achievements of these peoples contribute far more in all pre-Renaissance periods to the development of world science than has yet been realized."

The Shulba Sutras have preserved only that part of Vedic mathematics which was used for constructing the altars and for computing the calendar to regulate the performance of religious rituals. After the Shulba Sutra period, the main developments in Vedic mathematics arose from needs in the field of astronomy. The Jyotisha, science of the luminaries, utilizes all branches of mathematics.

The need to determine the right time for their religious rituals gave the first impetus for astronomical observations. With this desire in mind, the priests would spend night after night watching the advance of the moon through the circle of the nakshatras (lunar mansions), and day after day the alternate progress of the sun towards the north and the south. However, the priests were interested in mathematical rules only as far as they were of practical use. These truths were therefore expressed in the simplest and most practical manner. Elaborate proofs were not presented, nor were they desired.

Evolution of Arabic (Roman) Numerals from India

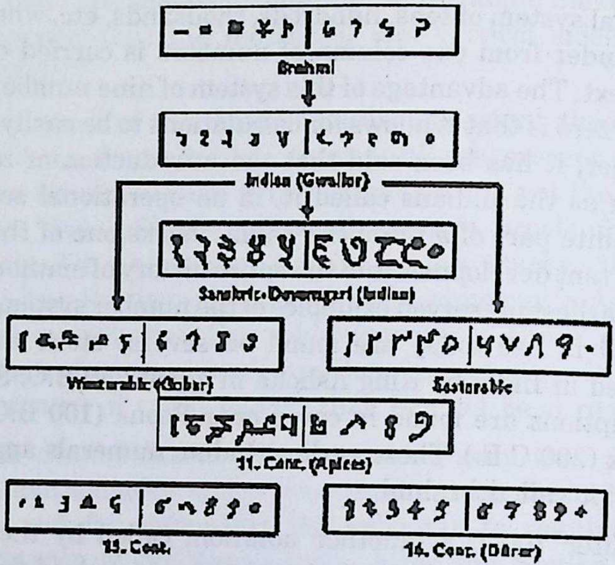
A close investigation of the Vedic system of mathematics shows that it was much more advanced than the mathematical systems of the civilizations of the Nile or the Euphrates. The Vedic mathematicians had developed the decimal system of tens, hundreds, thousands, etc. where the remainder from one column of numbers is carried over to the next. The advantage of this system of nine number signs and a zero is that it allows for calculations to be easily made. Further, it has been said that the introduction of zero, or sunya as the Indians called it, in an operational sense as a definite part of a number system, marks one of the most important developments in the entire history of mathematics. The earliest preserved examples of the number system which is still in use today are found on several stone columns erected in India by King Ashoka in about 250 B.C. Similar inscriptions are found in caves near Poona (100 B.C.) and Nasik (200 C.E.). These earliest Indian numerals appear in a script called brahmi.

After 700 C.E. another notation, called by the name "Indian numerals," which is said to have evolved from the brahmi numerals, assumed common usage, spreading to Arabia and from there around the world. When Arabic numerals (the name they had then become known by) came into common use throughout the Arabian empire, which extended from India to Spain, Europeans called them "Arabic notations," because they received them from the Arabians. However, the Arabians themselves called them "Indian figures" (Al-Arqan-Al-Hindu) and mathematics itself was called "the Indian art" (hindisat).

Evolution of "Arabic Numerals" from Brahmi (250 BC) to the 16th Century.

Mastery of this new mathematics allowed the Muslim mathematicians of Baghdad to fully utilize the geometrical treatises of Euclid and Archimedes. Trigonometry flourished there along with astronomy and geography. Later in history, Carl Friedrich Gauss, the "prince of mathematics," was said to have lamented that Archimedes in the third century B.C.

had failed to foresee the Indian system of numeration; how much more advanced science would have been.



Prior to these revolutionary discoveries, other world civilizations-the Egyptians, the Babylonians, the Romans, and the Chinese-all used independent symbols for each row of counting beads on the abacus, each requiring its own set of multiplication or addition tables. So cumbersome were these systems that mathematics was virtually at a standstill.

The new number system from the Indus Valley led a revolution in mathematics by setting it free. By 500 C.E. mathematicians of India had solved problems that baffled the world's greatest scholars of all time. Aryabhatta, an astronomer mathematician who flourished at the beginning of the 6th century, introduced sines and versed sines-a great improvement over the clumsy half-cords of Ptolemy. A.L. Basham, foremost authority on ancient India, writes in The Wonder That Was India, Medieval Indian mathematicians, such as Brahmagupta (seventh century), Mahavira (ninth century), and Bhaskara (twelfth century), made several discoveries which in Europe were not known until the Renaissance or later. They understood the import of positive and negative quantities, evolved sound systems of extracting square and cube roots, and could solve quadratic and certain types of indeterminate equations." Mahavira's most noteworthy contribution is his treatment of fractions for the first time and his rule for dividing one fraction by another, which did not appear in Europe until the 16th century.

Equations and Symbols

B.B. Dutta writes: "The use of symbols-letters of the alphabet to denote unknowns, and equations are the foundations of the science of algebra. The Hindus were the first to make systematic use of the letters of the alphabet to denote unknowns. They were also the first to classify and make a detailed study of equations.

Thus they may be said to have given birth to the modern science of algebra." The great Indian mathematician Bhaskaracharya (1150 C.E.) produced extensive treatises on both plane and spherical trigonometry and algebra, and his works contain remarkable solutions of problems which were not discovered in Europe until the seventeenth and eighteenth centuries. He preceded Newton by over 500 years in the discovery of the principles of differential calculus. A.L. Basham writes further, "The mathematical implications of zero (sunya) and infinity, never more than vaguely realized by classical authorities, were fully understood in medieval India. Earlier mathematicians had taught that X/0 = X, but Bhaskara proved the contrary. He also established mathematically what had been recognized in Indian theology at least a millennium earlier: that infinity, however divided, remains infinite, represented by the equation oo /X = oo." In the 14th century, Madhava, isolated in South India, developed a power series for the arc tangent function, apparently without the use of calculus, allowing the calculation of pi to any number of decimal places (since arctan 1 = pi/4). Whether he accomplished this by inventing a system as good as calculus or without the aid of calculus; either way it is astonishing.

Spiritually advanced cultures were not ignorant of the principles of mathematics, but they saw no necessity to explore those principles beyond that which was helpful in the advancement of God realization.

By the fifteenth century C.E. use of the new mathematical concepts from India had spread all over Europe to Britain, France, Germany, and Italy, among others. A.L. Basham states also that:

> *The debt of the Western world to India in this respect [the field of mathematics] cannot be overestimated. Most of the great discoveries and inventions of which Europe is so proud would have been impossible without a developed system of mathematics, and this in turn would have been impossible if Europe had been shackled by the unwieldy system of Roman numerals. The unknown man who devised the new system was, from the world's point of view, after the Buddha, the most important son of India. His achievement, though easily taken for granted, was the work of an analytical mind of the first order, and he deserves much more honour than he has so far received.*

Unfortunately, Eurocentrism has effectively concealed from the common man the fact that we owe much in the way of mathematics to ancient India. Reflection on this may cause modern man to consider more seriously the spiritual preoccupation of ancient India. The rishis (seers) were not men lacking in practical knowledge of the world, dwelling only in the realm of imagination. They were well developed in secular knowledge, yet only insofar as they felt it was necessary within a world view in which consciousness was held as primary.

In ancient India, mathematics served as a bridge between understanding material reality and the spiritual conception. Vedic mathematics differs profoundly from Greek mathematics in that knowledge for its own sake (for its aesthetic satisfaction) did not appeal to the Indian mind. The mathematics of the Vedas lacks the cold, clear, geometric precision of the West; rather, it is cloaked in the poetic language which so distinguishes the East.

Vedic mathematicians strongly felt that every discipline must have a purpose, and believed that the ultimate goal of life was to achieve self-realization and love of God and thereby be released from the cycle of birth and death.

Those practices which furthered this end either directly or indirectly were practiced most rigorously. Outside of the religio-astronomical sphere, only the problems of day to day life (such as purchasing and bartering) interested the Indian mathematicians.

Poetry in Math

One of the foremost exponents of Vedic math, the late Bharati Krishna Tirtha Maharaja, author of Vedic Mathematics, has offered a glimpse into the sophistication of Vedic math. Drawing from the Atharva-veda, Tirtha Maharaja points to many sutras (codes) or aphorisms which appear to apply to every branch of mathematics: arithmetic, algebra, geometry (plane and solid), trigonometry (plane and spherical), conics (geometrical and analytical), astronomy, calculus (differential and integral), etc.

Utilizing the techniques derived from these sutras, calculations can be done with incredible ease and simplicity in one's head in a fraction of the time required by modern means. Calculations normally requiring as many as a hundred steps can be done by the Vedic method in one single simple step. For instance the conversion of the fraction 1/29 to its equivalent recurring decimal notation normally involves 28 steps. Utilizing the Vedic method it can be calculated in one simple step.

In order to illustrate how secular and spiritual life were intertwined in Vedic India, Tirtha Maharaja has demonstrated that mathematical formulas and laws were often taught within the context of spiritual expression (mantra). Thus while learning spiritual lessons, one could also learn mathematical rules.

Tirtha Maharaja has pointed out that Vedic mathematicians prefer to use the devanagari letters of Sanskrit to represent the various numbers in their numerical notations rather than the numbers themselves, especially where large numbers are concerned. This made it much easier for the students of this math in their recording of the arguments and the appropriate conclusions.

Tirtha Maharaja states, "In order to help the pupil to memorize the material studied and assimilated, they made it a general rule of practice to write even the most technical and abstruse textbooks in sutras or in verse (which is so much easier-even for the children-to memorize).

And this is why we find not only theological, philosophical, medical, astronomical, and other such treatises, but even huge dictionaries in Sanskrit verse! So from this standpoint, they used verse, sutras and codes for lightening the burden and facilitating the work (by versifying scientific and even mathematical material in a readily assimilable form)!" The code used is as follows:

The Sanskrit consonants
ka, ta, pa, and ya all denote 1;
kha, tha, pha, and ra all represent 2;
ga, da, ba, and la all stand for 3;
Gha, dha, bha, and va all represent 4;
gna, na, ma, and sa all represent 5;

ca, ta, and sa all stand for 6;
cha, tha, and sa all denote 7;
ja, da, and ha all represent 8;
jha and dha stand for 9; and
ka means zero.

Vowels make no difference and it is left to the author to select a particular consonant or vowel at each step. This great latitude allows one to bring about additional meanings of his own choice.

For example kapa, tapa, papa, and yapa all mean 11. By a particular choice of consonants and vowels one can compose a poetic hymn with double or triple meanings. Here is an actual sutra of spiritual content, as well as secular mathematical significance.

gopi bhagya madhuvrata

srngiso dadhi sandhiga

khala jivita khatava

gala hala rasandara

While this verse is a type of petition to Krishna, when learning it one can also learn the value of pi/10 (i.e. the ratio of the circumference of a circle to its diameter divided by 10) to 32 decimal places. It has a self-contained master-key for extending the evaluation to any number of decimal places.

The translation is as follows:

O Lord anointed with the yogurt of the milkmaids'
worship (Krishna), O savior of the fallen, O master
of Shiva, please protect me.

At the same time, by application of the consonant code given above, this verse directly yields the decimal equivalent of pi divided by 10:

pi/10 = 0.31415926535897932384626433832792.

Thus, while offering mantric praise to Godhead in devotion, by this method one can also add to memory significant secular truths.

This is the real gist of the Vedic world view regarding the culture of knowledge: while culturing transcendental knowledge, one can also come to understand the intricacies of the phenomenal world. By the process of knowing the absolute truth, all relative truths also become known. In modern society today it is often contended that never the twain shall meet: science and religion are at odds. This erroneous conclusion is based on little understanding of either discipline. Science is the smaller circle within the larger circle of religion.

We should never lose sight of our spiritual goals. We should never succumb to the shortsightedness of attempting to exploit the inherent power in the principles of mathematics or any of the natural sciences for ungodly purposes. Our reasoning faculty is but a gracious gift of Godhead intended for divine purposes, and not those of our own design.

Bibliography

A B Chace, L S Bull, H P Manning and R C Archibald: *The Rhind Mathematical Papyrus,* Oberlin, Ohio, 1927.

A.L. Basham: *The Wonder That Was India,* Rupa & Co., Calcutta, 1967.

Aaboe, Asger: *Episodes from the Early History of Mathematics*, New York, 1964.

Amma, Saraswarti: *Geometry in Ancient and Medieval India*, Motilal Banarsidas, 1979.

Armstrong, Robert G.: *Yoruba Numerals*, London, Oxford University Press, 1962.

Arnett, William S.: *The Predynastic Origin of Egyptian Hieroglyphs*, Washington, University Press of America, 1982.

Ascher, Marcia: *Ethnomathematics*, Pacific Grove, CA: Brooks/Cole, 1991.

Bag, A. K.: *Mathematics in Ancient and Medieval India*, Varanasi: Chaukhambha Orientalia, 1979.

Barnard, F. P.: *The Casting-Counter and the Counting Board. A Chapter in the History of Numismatics and Early Arithmetic*, Oxford, Oxford University Press, 1916.

Barrow, John D.: *Pi in the Sky*, Oxford, Clarendon Press, 1992.

Basham, A. L.: *The Wonder That Was India*, Calcutta, Rupa & Co., 1967.

Beckmann, Petr: *A History of Pi*, New York, St. Martin's Press, 1974.

Bell, Eric T.: *The Magic of Numbers*, New York, McGraw-Hill, 1946.

Berlin, Brent: Tzeltal Numeral Classifiers, The Hague, Mouton, 1968.

Bharati Krishna Tirthaji Majaraja.: *Vedic Mathematics*. Delhi, Motilal Banarsidass, 1988.

Bidyaranya, Swami: *History of Hindu Mathematics*,Bombay, Asia Pub. House, 1962.

Bischoff, Bernard: *Latin Palaeography*, Cambridge, Cambridge University Press, 1990.

Bonfante, Larissa: *Etruscan*. Berkeley, University of California Press, 1990.

Bowditch, Charles P.: *The Numeration, Calendar Systems and Astronomical knowledge of the Mayas*, Cambridge, Cambridge University Press, 1910.

Brainerd, Charles J.: *The Origins of the Number Concept*, New York, Praeger, 1979.

Brunschwig, Jacques and Geoffrey E.R. Lloyd:. *Greek Thought: A Guide to Classical Knowledge*, Cambridge, Harvard University Press, 2000.

Bunt, L. N. H., Jones, P. S., & Bedient, J. D.: *The Historical Roots of Elementary Mathematics*, New York, Dover, 1988.

Chattopadhyaya, D. P.: *Mathematics, Astronomy, and Biology in Indian Tradition: Some Conceptual Preliminaries*, New Delhi, Project of History of Indian Science, Philosophy, and Culture, 1995.

Clawson, Calvin: *The Mathematical Traveller*, New York, Plenum Press, 1994.

Closs, Michael P.: *Native American Mathematics*, Austin, University of Texas Press, 1986.

Conant, Levi L.: *The Number Concept*, New York, MacMillan, 1896.

Crossley, John N: *The Emergence of Number*, Singapore, World Scientific, 1987.

Crump, Thomas: *The Anthropology of Numbers*; Cambridge, Cambridge University Press, 1990.

Dantzig, Tobias: *Number, the Language of Science*, London, George Allen and Unwin, 1940.

De Villiers, Melius: *The Numeral-Words: Their Origin, Meaning, History and Lesson*, London, Witherby, 1923.

Dehaene, Stanislas: *The Number Sense*, New York, Oxford University Press, 1997.

Dilke, Oswald Ashton Wentworth: *Mathematics and Measurement*, Berkeley, CA: U. of Cal. Press, 1987.

Eves, Howard: *An Introduction to the History of Mathematics*, New York, Rinehart and Company Inc., 1953.

Flegg, Graham: *Numbers: Their History and Meaning*, London, André Deutsch, 1983.

Fowler, David: *The Mathematics of Plato's Academy*, Oxford, Clarendon, 1999.

Fuson, K.C.: *Children's Counting and Concepts of Number*, New York, Springer-Verlag, 1988.

Ganeri, Anita: *The Story of Numbers and Counting*, New York, Oxford University Press, 1996.

Gokhale, Shebhana Laxman: *Indian Numerals*, Poona, Deccan College, 1966.

Griffith, Francis Llewellyn: *Hieratic Papyri from Kahun and Gurob*, London, Bernard Quaritch, 1898.

Grover, Sudarshan: *History of Development of Mathematics in India*, Delhi, Atma Ram & Sons, 1994.

Haldane, John Burdon Sanderson: *Science and Indian Culture*, Calcutta, New Age Publishers, 1965.

Halliwell, James Orchard: *Rara Mathematica*, London, J.W. Parker, 1839.

Herbert Meschkowski: *Ways of Thought of Great Mathematicians*, Holden-Day Inc., San Francisco, 1964.

Hogben, Lancelot: *Mathematics for the Million*, New York, W. W. Norton, 1937.

Howard Eves: *An Introduction to the History of Mathematics*, Rinehart and Company Inc., New York, 1953.

Ifrah, Georges: *From One to Zero: A Universal History of Numbers*, New York, Viking Penguin, 1985.

Jagadguru Swami Shri Bharati Krishna Tirthaji Maharaja: *Vedic Mathematics*, Motilal Banarsidass, Delhi, 1988.

Jaggi, O. P.: *Science and Technology in Medieval India*, New Delhi, Atma Ram & Sons, 1977.

Kline, Morris: *Mathematical Thought from Ancient to Modern Times*, New York, Oxford U. Press, 1990.

Kuppuram, G & K. Kumudamani: *History of Science and Technology in India*, Delhi, Sundeep Prakashan, 1990.

Lancy, David F.: *Cross-Cultural Studies in Cognition and Mathematics*, New York, Academic, 1983.

Marcus, Joyce: *Mesoamerican Writing Systems*, Princeton, Princeton University Press, 1992.

Marshack, Alexander: *The Roots of Civilization*, New York, McGraw-Hill, 1972.

Martzloff, Jean-Claude: *A History of Chinese Mathematics*, New York, Springer, 1997.

McLeish, John: *The Story of Numbers*, New York, Fawcett Columbine, 1991.

Menninger, Karl: *Number-Words and Number-Symbols*. London, MIT, 1969.

Murthy, T. S. Bhanu: *A Modern Introduction to Ancient Indian Mathematics*, Wiley Eastern Ltd., New Delhi, 1992.

Neugebauer, Otto and A. Sachs: *Mathematical Cuneiform Texts*, New Haven, Yale University Press, 1945.

Neugebauer, Otto: *The Exact Sciences in Antiquity*, Princeton, Princeton University Press, 1952.

North, John: *The Norton History of Astronomy and Cosmology*, New York, W.W.Norton, 1995.

Pandit, M. D.: *Mathematics as known to the Vedic Samhitas*, Delhi, Sri Satguru Publications, 1993.

Pappas, Theoni: *The Magic of Mathematics*, San Carlos, Wide World Publishing / Tetra, 1994.

Parpola, Asko: *Deciphering the Indus Script*, Cambridge, Cambridge University Press, 1994.

Peet, T. Eric: *The Rhind Mathematical Papyrus*, London, British Museum, 1923.

Piaget, Jean: *The Child's Conception of Number*, New York, W.W. Norton, 1965.

Pullan, J.M: *The History of the Abacus*, New York, Praeger, 1968.

Rao, S. Balaachandra: *Indian Mathematics and Astronomy*, Bangalore, Jnana Deep Publications, 1994.

Restivo, Sal: *The Social Relations of Physics, Mysticism and Mathematics*, Dordrecht, Kluwer, 1983.

Robins, G. and C. Shute: *The Rhind Mathematical Papyrus*, London, British Museum, 1987.

Russell, Bertrand: *The Principles of Mathematics*, Cambridge, Cambridge University Press, 1903.

Salomon, Richard: *Indian Epigraphy*, New York, Oxford University Press, 1998.

Saraswati Amma: *Geometry in Ancient and Medieval India*, Motilal Banarsidas, 1979.

Sarton, George: *The Study of the History of Mathematics*, Cambridge, Cambridge University Press, 1936.

Satterthwaite, Linton, Jr.: *Concepts and Structures of Maya Calendrical Arithmetics*, Philadelphia, University of Pennsylvania, 1947.

Schaaf, W.L.: *Mathematics, Our Great Heritage*, New York, Harper and Row, 1948.

Schimmel, Annemarie: *The Mystery of Numbers*, New York, Oxford University Press, 1993.

Scriba, C.J.: *The Concept of Number*, Zurich, Mannheim, 1968.

Seidenberg, A.: *The Zero in the Mayan Numerical Notation*, Austin, University of Texas Press, 1986.

Smeltzer, Donald: *Man and Number*, New York, Emerson, 1959.

Smith, D. E.: *History of Mathematics*, New York, Dover Publications Inc., 1951.

Srinivasiengar, C. N.: *The History of Ancient Indian Mathematics*. Calcutta, World Press, 1967.

Struik, Dirk J.: *A Concise History of Mathematics*, New York, Dover, 1987.

Thompson, Sir J. Eric S.: *Maya Hieroglyphic Writing: An Introduction*, Norman, University of Oklahoma Press, 1971.

Todd, R.R., P.J. Barber, and D. Jones: *The Internal Representation of Number: Analogue or Digital?*, Oxford, Clarendon, 1987.

Urton, Gary: *The Social Life of Numbers*, Austin, University of Texas Press, 1997.

Volodarsky, A.I.: *Mathematics in Ancient India*, Tokyo, Science Council of Japan, 1974.

Woolner, Alfred C.: *Introduction to Prakrit*, Calcutta, Baptist Mission Press, 1917.

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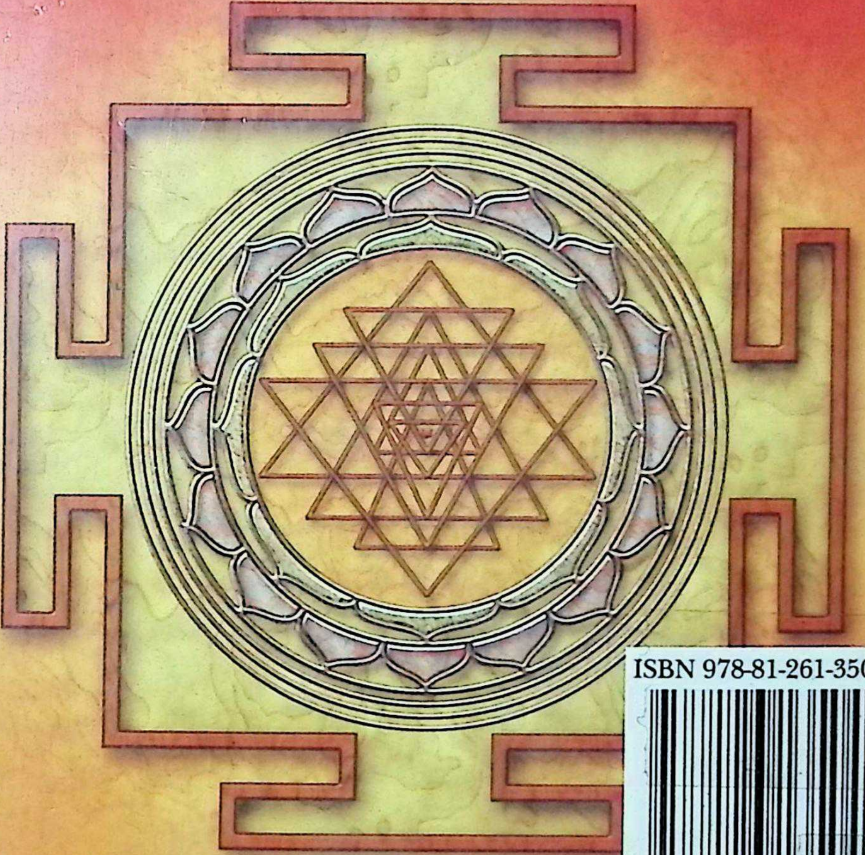

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